DET
Department of Electronics and Telecommunications

Averaging inequalities.
A new tool for multi-agent systems analysis

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Plan of the talk

Motivation

- Distributed algorithms intersecting convex sets: constrained/optimal consensus

Averaging inequalities: consensus and convergence

- General considerations;
- Static graph: consensus criterion;
- Dynamic graph: reciprocity conditions.

Applications

- Constrained consensus revisited: a common fixed point problem
- Opinion dynamics models;

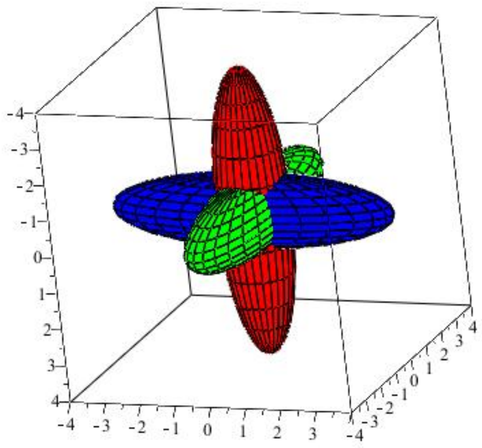
Conclusions. Future works

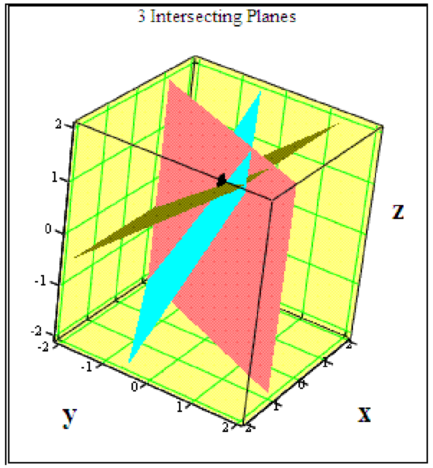
Plan of the talk

Motivation

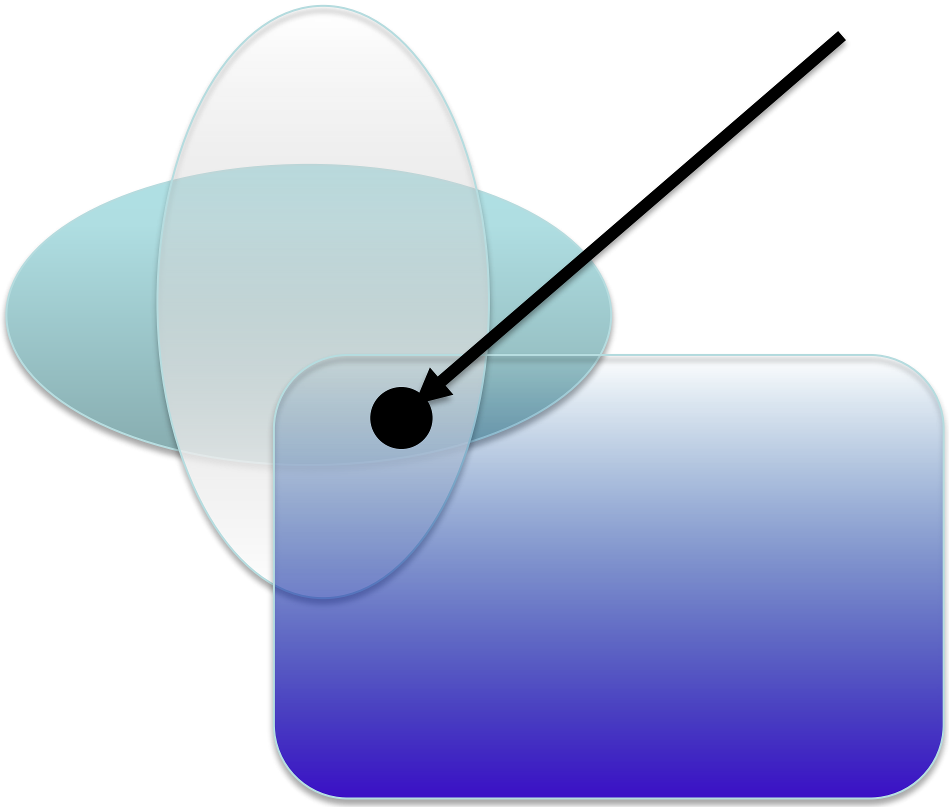
- Distributed algorithms intersecting convex sets: constrained/optimal consensus

Intersection of convex sets: a classical problem in applied mathematics





Very difficult to visualize in 3D…



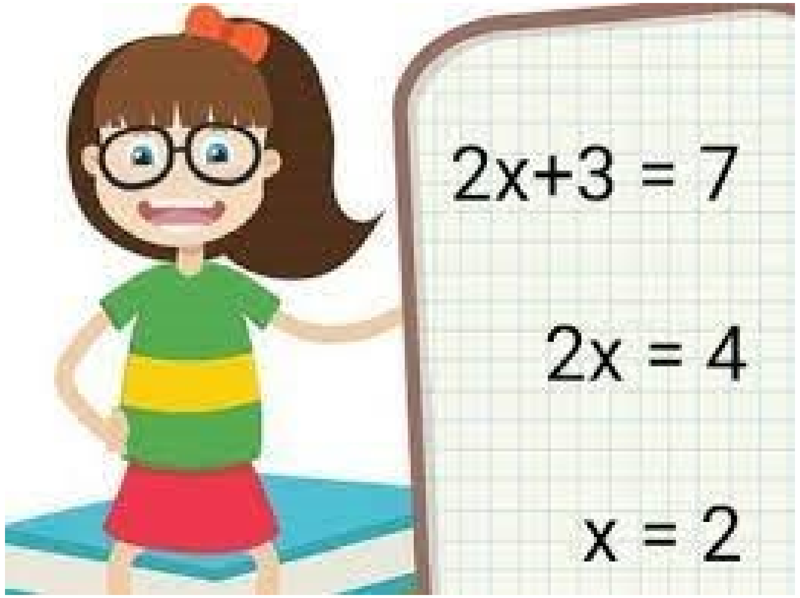
Not easy to describe even in 2D…

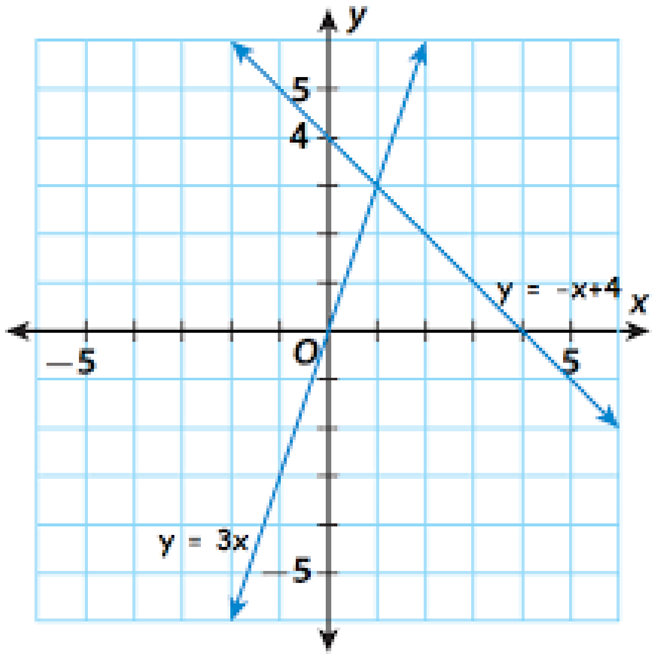
Plenty of applications in numerical analysis, optimization, data science etc.

To describe the whole intersection is difficult, let’s find at least one point (assumed to exist).

Example from the school: linear equations (= intersection of hyperplanes)

$$\mathrm{Ax} = \mathrm{b}$$



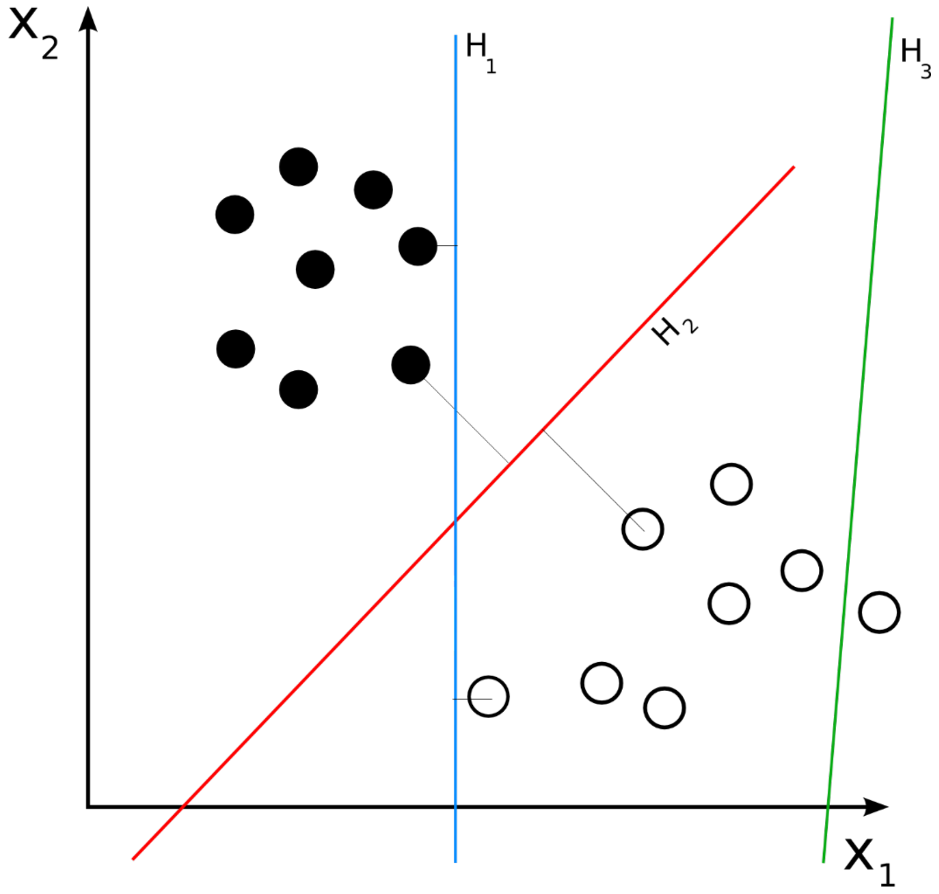


May be considered as a problem of intersecting several hyperplanes.

Becomes non-trivial as the dimension becomes huge (e.g., for PageRank or SJR rank computation, Leontief balance equations in economics etc).
Special methods have been developed, many of them exploit special structure of the matrix.

Linear classifier problem (= intersection of half-spaces, solving inequalities)

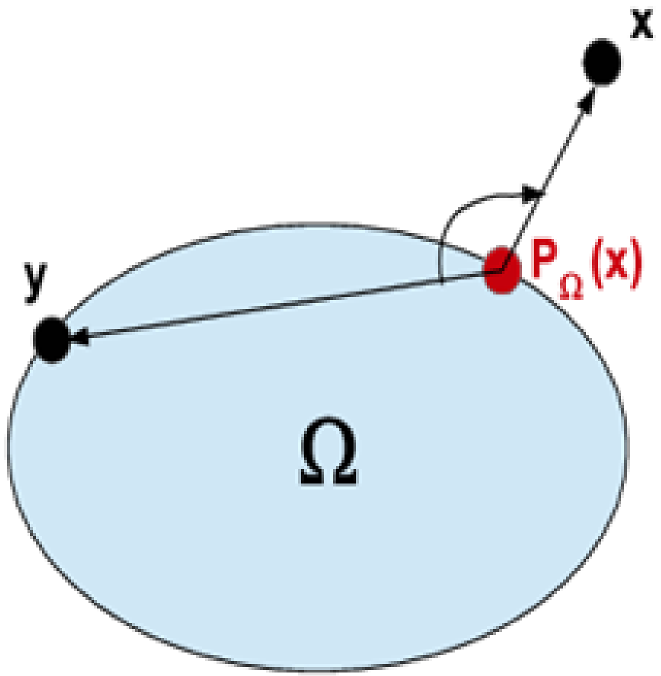
$$\mathrm{Ax} \leq \mathrm{b}$$



Find a hyperplane separating two clouds of points (red, blue –ok, green – not good)

Other problems: intersections of strips, ellipsoids, other “nonlinear” convex sets, see e.g.
Combettes, The foundations of set-theoretic estimation//Proceedings of IEEE, 1993

Projection onto convex closed sets



Projection operator:

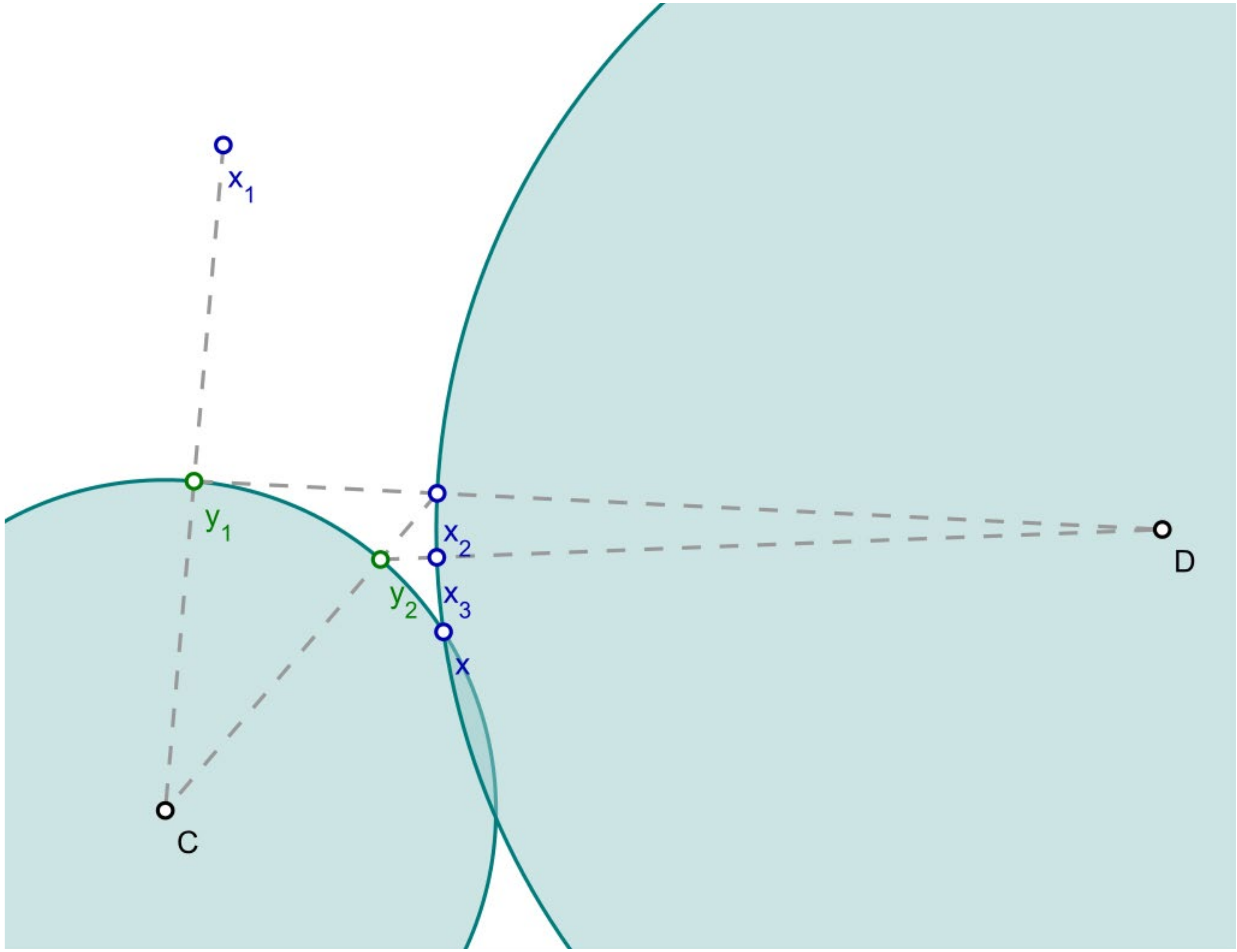
returns the nearest point of a convex closed set

$$\forall y \in \Omega \;\; |x-y|^2 \geq |x-P_\Omega(x)|^2 + |y-P_\Omega(x)|^2$$

$$\Omega = \{y : a^\top y = b\} \implies P_\Omega(x) = x + \frac{b - a^\top x}{|a|^2} a$$

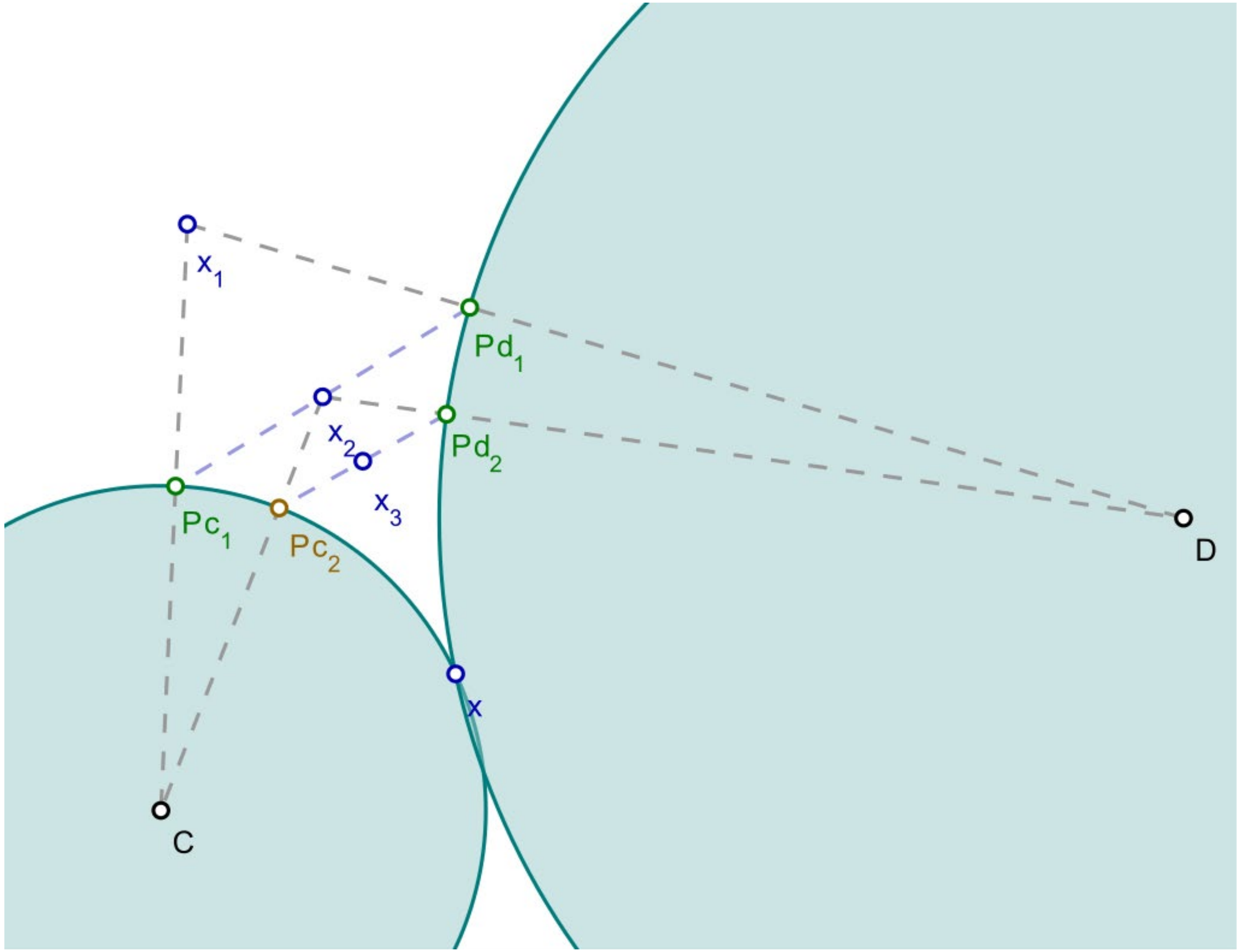
The method of alternating projections, or POCS (projection onto convex sets)

Illustration for two sets, the method works similarly for any finite number of sets



A modification of POCS: averaged projections

Illustration for two sets, the method works similarly for any finite number of sets



Some history: dates back to 1930s (mainly hyperplanes/halfspaces), reopened many times since then.

FIRST WORKS:

- J. von Neumann, Functional Operators, Vol. II. The Geometry of Orthogonal Spaces, Princeton Univ. Press, 1933 (reprinted in 1950)
- Kaczmarz S. Angenäherte Auflösung von Systemen linearer Gleichungen// Bull. Int. l'Acad. Polon. Sci. Lett. A. — 1937. — 35. — C. 355–357;
- Cimmino G. Calcolo approssiomatto per le soluzioni dei sistemi di equazioni lineari// La Ricerca Sci. XVI. Ser. II. — 1938. — 1. — C. 326–333.

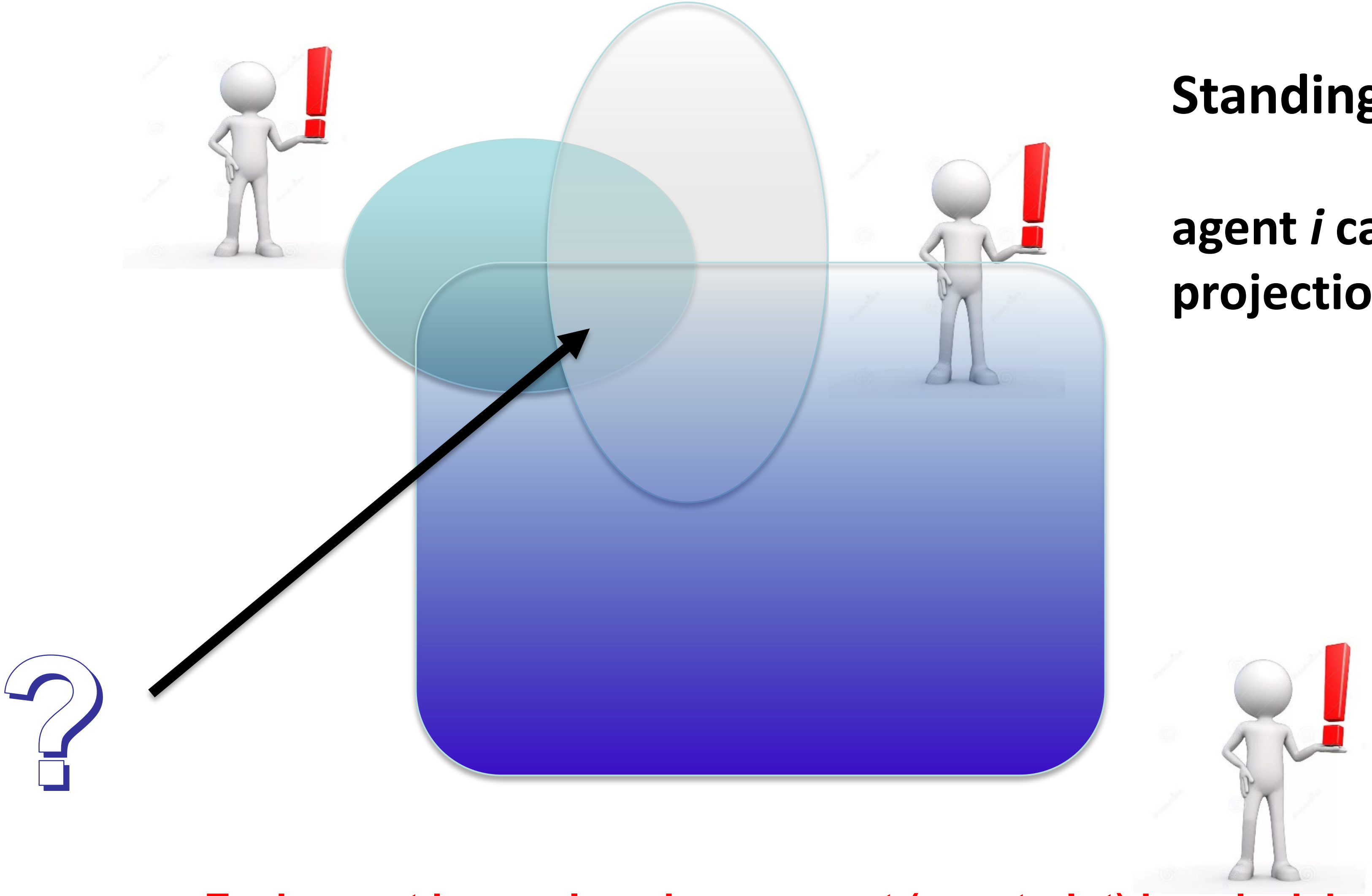
SIMILAR IN SPIRIT METHODS and/or REDISCOVERIES:

- Perceptron learning algorithms (1950s);
- Bregman relaxation method (1966);
- Gubin-Polyak-Raik (1967);
- Yakubovich method of recursive objective inequalities (1968).

WE ARE INTERESTED IN DISTRIBUTED MODIFICATIONS.

Constrained (optimal) consensus:

Intersecting sets that belong to independent agents



Standing assumption:

agent *i* can compute P_i -- the projection onto its constraint Ξ_i

Each agent has a closed convex set (constraint) in a decision space.

Find a point in the decision space, satisfying all of these constraints

Constrained (optimal) consensus:

Distributed algorithms available in the literature.

$$\xi^i(t+1) = P_i\left[\sum\nolimits_{j\in\mathcal{V}} w_{ij}(t)\xi^j(t)\right]\ \forall i \in \mathcal{V}$$

Nedic, Ozdaglar, Parrilo (2010)

$$\xi^i(t+1) = P_i\left[\sum\nolimits_{j\in\mathcal{V}}^{n} w_{ij}(t)P_j(\xi^j(t))\right]\ \forall i \in \mathcal{V}$$

Liu, Morse, Nedic, Basar (2014)

$$\xi^i(t+1) = w_{ii}(t)P_i(\xi^i(k)) + \sum\nolimits_{j\neq i} w_{ij}(t)\xi^j(t)\,\forall i \in \mathcal{V}.$$

You, Song, Tempo (2016)

*[Weight matrix **W** is stochastic]*

$$\dot{\xi}^i(t) = \sum_{j\in\mathcal{V}} a_{ij}(t)(\xi^j(t) - \xi^i(t)) + P_i(\xi^i(t)) - \xi^i(t).$$

Shi, Johansson, Hong (2013)

*[Weight matrix **A** is nonnegative]*

Constrained consensus vs consensus policies

Look like usual consensus protocols, but actually the properties are **quite different**:

- The **solutions are bounded**, but the diameter of the convex hull spanned by the states, generally, does not decrease. The convergence is examined by means of a special Lyapunov function: (squared) distance to the desired set (or some point in the desired set)

$$\left\{ x : x_i = x_j \in \Xi_* = \bigcap_{k=1}^{n} \Xi_k \quad \forall i \right\}$$

- The set of equilibria is narrower that in usual consensus algorithms:

- Conditions for consensus are different, e.g., in the static graph case we need **strong connectivity** instead of a **spanning tree existence**;

Why does these algorithm work? For two reasons

$$\xi^i(k+1) = w_{ii}(k)P_i(\xi^i(k)) + \sum_{j\neq i} w_{ij}(k)\xi^j(k)\,\forall i \in \mathcal{V}.$$

The first key observation: projection operators are **paracontractions.**

Map $M : \mathbb{R}^m \to \mathbb{R}^m$ **is a paracontraction if it is continuous and**

$$\|M(\xi) - \xi_0\| < \|\xi - \xi_0\| \;\; \forall \xi \notin \mathcal{F}(M),\, \xi_0 \in \mathcal{F}(M),$$
$$\mathcal{F}(M) = \{z \in \mathbb{R}^m : M(z) = z\}.$$

The second key observation: **a hidden system of one-sided inequalities**

$$\xi_* \in \Xi_*, \quad x_i(k) = |\xi_i(k) - \xi_*| \implies$$
$$0 \leq x_i(k+1) \leq w_{ii}|P_i(\xi_i(k)) - \xi_*| + \sum_{j\neq i} w_{ij}|\xi_j(k) - \xi_*| \leq$$
$$\leq \sum_{j\neq i} w_{ij}x_j(k) + w_{ii}|\xi_i(k) - \xi_*| = \sum_j w_{ij}x_j(k) \quad \forall i.$$

We obtain a novel mathematical object:
The system of averaging inequalities associated to a consensus protocol

$$x(k+1) = Wx(k) \qquad \dot{x}(t) = -L[A]x(t)$$

$$x(k+1) \le Wx(k) \qquad \dot{x}(t) \le -L[A]x(t)$$

*[Weight matrix **W** is stochastic]* *[Weight matrix **A** is nonnegative]*

May be considered from two viewpoints:

- **as recurrent or differential inclusions (with unbounded right-hand side);**
- **as averaging algorithms with an unknown sign-preserving excitation**

$$x(k+1) = Wx(k) + \eta(k), \quad \eta_i(k) \le 0 \quad \forall i \forall k.$$

In both situations, however, the existing results do not help much to study the solutions! Neither the theory of differential inequalities in ODE theory helps.

We need a special theory.

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Averaging inequalities: consensus and convergence

- General considerations;
- Static graph: consensus criterion;
- Dynamic graph: reciprocity conditions.

General considerations

$$x(k+1) \leq W(k)x(k) \iff x_i(k+1) \leq \sum_j w_{ij}(k)x_j(k) \quad \forall i,$$

$$\dot{x}(t) \leq -L[A(t)]x(t) \iff \dot{x}_i(t) \leq \sum_j a_{ij}(t)(x_j(t) - x_i(t))$$

- **Solutions semi-bounded from above, maximal component non-increasing**

$$\max x(t) \leq \max x(t') \quad \forall t \geq t' \geq 0$$

- **We are primarily interested in solutions that admit a-priori lower bound (as in the previous example: the vector of distances is nonnegative by construction).**

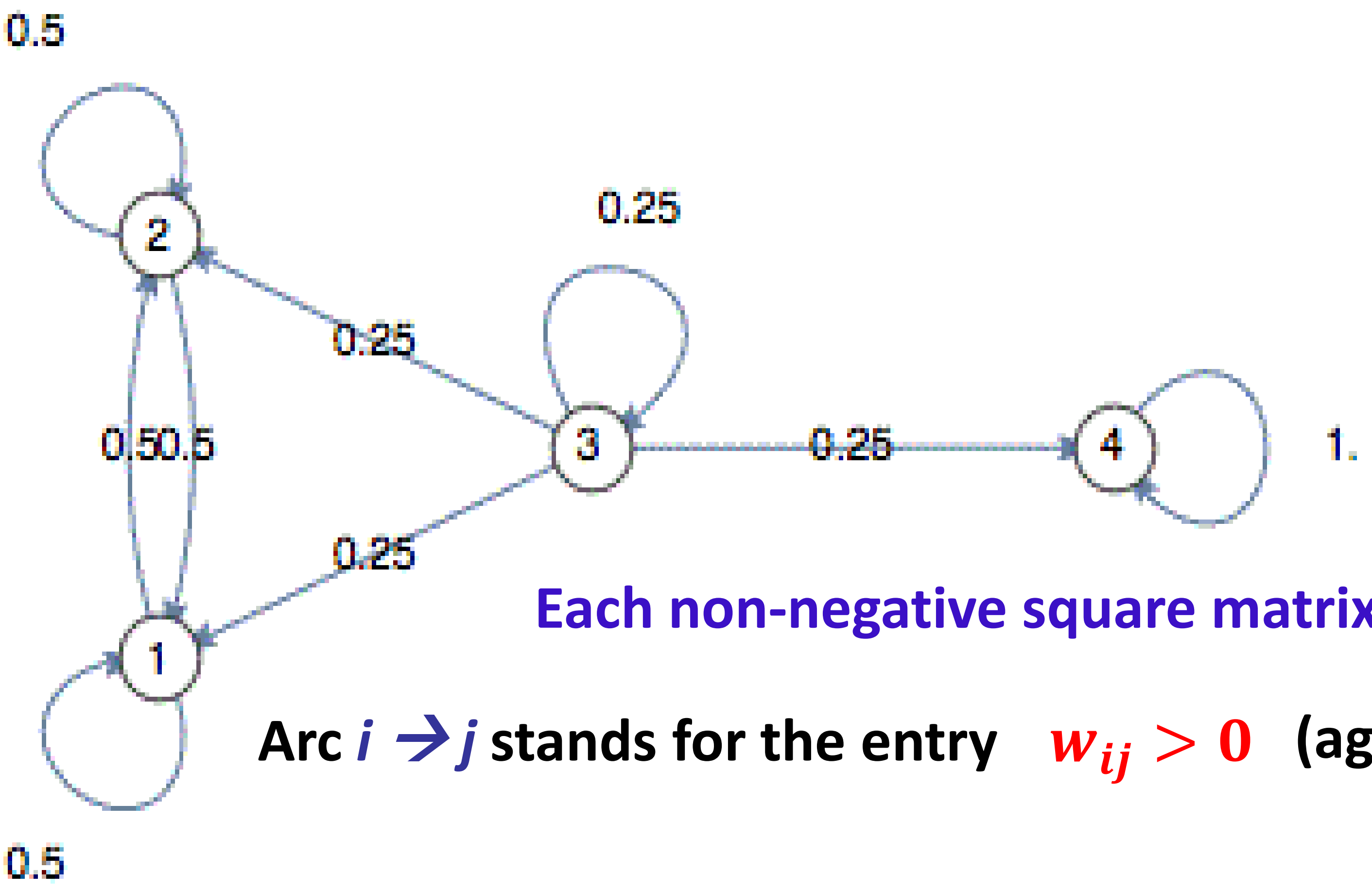
- **Generally, solutions can behave very irregularly, e.g., consider a trivial 2-agent example**

$$x_1(k) = (-1)^k \xi_k, \; x_2(k) \equiv 4, \quad \xi_k \in \{0, \pm 1\},$$

$$x_1(k+1) \leq \frac{1}{2}x_1(k) + \frac{1}{2}x_2(k), \; x_2(k+1) \leq 0x_1(k) + 1x_2(k)$$

- **Surprisingly, if the graph is strongly connected, solutions do converge to consensus!**

Preliminaries: the graph of a nonnegative matrix



$$W = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Each non-negative square matrix determines a weighted graph

Arc *i* → *j* stands for the entry $w_{ij} > 0$ **(agent *i* is influenced by agent *j*)**

A cut in the graph: splitting of all nodes into two disjoint sets

$$I \cup J = \{1, 2, \ldots, N\}, \quad I \cap J = \emptyset$$

Inequalities with a constant matrix

Theorem (P., Cao, 2017)

Let *W* be a square row-stochastic irreducible matrix with positive diagonal entries.
Then, each solution to the system of recurrent inequalities

$$x(k+1) \leq Wx(k) \iff x_i(k+1) \leq \sum_j w_{ij}x_j(k)\,\forall i$$

converges to a vector of identical components $x(k) \xrightarrow[k\to\infty]{} \xi_* \mathbf{1}_N, \quad \xi_* \in \mathbb{R} \cup \{-\infty\}$

If, additionally, $\xi_* > -\infty$ **then the ``residual" between left and right-hand sides vanishes**

$$Wx(k) - x(k+1) \xrightarrow[k\to\infty]{} 0$$

- The requirement of irreducibility (strong connectivity of the graph) is in fact also **necessary** for convergence of all solutions to consensus (**different** from the usual consensus algorithm!)
- The result applies, obviously to the reversed inequalities

$$x(k+1) \geq Wx(k) \quad (\xi_* \in \mathbb{R} \cup \{+\infty\})$$

- If the graph has **isolated** aperiodic strongly connected components, the theorem guarantees clustering: the subvectors corresponding to the components reach consensus.

Sketch of the proof

$$x(k+1) \le Wx(k) \iff x_i(k+1) \le \sum_j w_{ij} x_j(k)\, \forall i$$

Important constructions: ordering permutation of the vector and the minimal positive entry

$$\max_i x_i(k) = y_1(k) \ge y_2(k) \ge \ldots \ge y_N(k) = \min_i x_i(k)$$

$$q := \min\{w_{ij} : w_{ij} > 0\}.$$

$w_{ii} \geq q$

Observation 1: the maximum is not increasing $y_1(k+1) \le y_1(k) \implies \exists \xi_* = \lim_{k\to\infty} y_1(k)$

Observation 2: $w_{ij} > 0 \implies x_i(k+1) \le q x_j(k) + (1-q) y_1(k)$ (*)

$$\forall i \;\; w_{ii} > 0 \implies x_i(k+1) \le q x_i(k) + (1-q) y_1(k) \qquad (!)$$

Observation 3: $y_1(k+1) \le q y_2(k) + (1-q) y_1(k) \implies y_2(k) \xrightarrow[k\to\infty]{} \xi_*.$

$$\exists i_1 = i_1(k), j \ne i_1 : y_1(k) = x_{i_1}(k), w_{i_1 j} > 0 \overset{(*)}{\implies} x_{i_1}(k+1) \le qy_2(k) + (1-q)y_1(k)$$

$$\forall m \ne i_1 : x_m(k) \le y_2(k) \overset{(!)}{\implies} x_m(k+1) \le q y_2(k) + (1-q) y_1(k).$$

Iterating the procedure: $y_s(k+1) \le q y_{s+1}(k) + (1-q) y_1(k)$

$$y_1(k), \ldots, y_s(k) \xrightarrow[k\to\infty]{} \xi_* \implies y_{s+1}(k) \xrightarrow[k\to\infty]{} \xi_*.$$

Inequalities with a time-varying matrix: reciprocity conditions

Theorem (P., Calafiore, Cao, 2020)

Let *W(k)* be stochastic matrices with strictly positive diagonal entries $w_{ii}(k) \geq \delta > 0 \ \ \forall i, k$

Assume also that nonzero off-diagonal entries are also strictly positive $w_{ij}(k) \in \{0\} \cup [\delta, 1]$

Suppose also that for each cut *(I,J)* in the graph the reciprocity condition holds:

$$\exists i \in I, j \in J, k \geq 0\colon \ w_{ij}(k) > 0 \Rightarrow \exists j' \in J, i' \in I, 0 \leq s \leq L\colon w_{j'i'}(k+s) > 0$$

Then, each solution to the system of recurrent inequalities

$$x(k+1) \leq W(k)x(k) \iff x_i(k+1) \leq \sum_j w_{ij} x_j(k) \, \forall i$$

converges

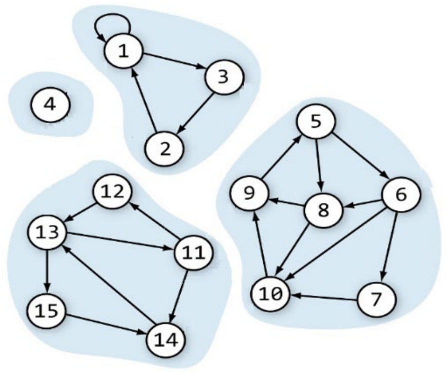
$$x(k) \xrightarrow[k\to\infty]{} x^\infty = (x_1^\infty, \ldots, x_N^\infty)^\top, \ x_j^\infty \geq -\infty.$$

Furthermore, two agents *i* and *j eventually reach consensus* $x_i^\infty = x_j^\infty$ **if they interact persistently** $\sum_{k=0}^{\infty} w_{ij}(k) = \infty$ **[global consensus if the persistent graph is strong]**

For bounded solutions, the residual vanishes $x(k+1) - W(k)x(k) \xrightarrow{k\to\infty} 0.$

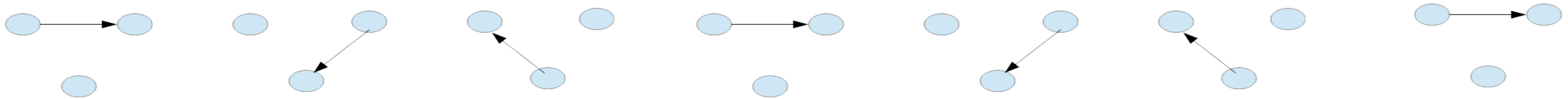
Main examples of reciprocal weight matrices

- **Static matrix *W:* the graph has isolated strongly connected components**



- **General *W(k) [*positive diagonal and strict positivity of nonzero entries]:**

the sequence of graphs is repeatedly strongly connected with some period



- **Type-symmetric *W(k) [*positive diagonal and strict positivity of nonzero entries, can be further relaxed]: the graphs are bidirectional, influence is mutual**

$$C^{-1}w_{ji}(k) \leq w_{ij}(k) \leq Cw_{ji}(k)$$

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- Opinion dynamics models;

Conclusions. Future works

Let us have another look at constrained consensus algorithms:

In reality, the nonlinear operators need not be projections.

$$\xi^i(t+1) = P_i\left[\sum_{j\in\mathcal{V}} w_{ij}(t)\xi^j(t)\right] \forall i \in \mathcal{V}$$

Nedic, Ozdaglar, Parrilo (2010)

$$\xi^i(t+1) = P_i\left[\sum_{j\in\mathcal{V}}^{n} w_{ij}(t)P_j(\xi^j(t))\right] \forall i \in \mathcal{V}$$

Liu, Morse, Nedic, Basar (2014)

$$\xi^i(t+1) = w_{ii}(t)P_i(\xi^i(k)) + \sum_{j\neq i} w_{ij}(t)\xi^j(t) \,\forall i \in \mathcal{V}.$$

You, Song, Tempo (2016)

Instead of projections, consider paracontractions that have a common fixed point:

$$\Xi_i = \{x : P_i(x) = x\} \neq \emptyset, \quad \Xi_* = \bigcap_{i=1}^{N} \Xi_i \neq \emptyset.$$

$$\forall x \notin \Xi_i \,\forall y \in \Xi_i \quad |P_i(x) - P_i(y)| = |P_i(x) - y| < |x - y|.$$

Fullmer, Liu, Morse, Proc. of IEEE CDC 2016, Proc. Of ACC 2017, IEEE TAC, 2017

Why do the algorithms work? Consider the simplest one!

Let W be static aperiodic irreducible matrix and the agents apply the algorithm

$$\xi_i(k+1) = w_{ii}P_i(\xi_i(k)) + \sum_{j\neq i} w_{ij}\xi_j(k).$$

Observation 1. The algorithm hides the inequality inside that admits a nonnegative solution

$$\xi_* \in \Xi_*, \quad x_i(k) = \|\xi_i(k) - \xi_*\| \implies$$

$$0 \le x_i(k+1) \le w_{ii}\|P_i(\xi_i(k)) - \xi_*\| + \sum_{j\neq i} w_{ij}\|\xi_j(k) - \xi_*\| \le \sum_j w_{ij}x_j(k).$$

Observation 2. We know that (bounded) solutions have finite limits and the "residuals" asymptotically vanish, which (omitting some boring details) means that the algorithms become "almost" linear as time grows

$$x_i(k) - \|P_i(\xi_i) - \xi_*\| \xrightarrow[k\to\infty]{} 0 \implies \varepsilon_i(k) = \xi_i(k) - P_i(\xi_i(k)) \to 0;$$

$$\xi_i(k+1) = \sum_j w_{ij}\xi_j(k) + \varepsilon_i(k),$$

Observation 3. Consensus algorithms are robust against disturbances $|\xi_i(k) - \xi_j(k)| \xrightarrow[k\to\infty]{} 0$

Using boundedness of solutions, all agents' vectors converge to the desired set.

$$\max_j d(\xi_i(k), \Xi_j) \to 0 \implies d(\xi_i(k), \Xi_*) = 0 \forall i$$

Example:

Simultaneous Optimization of Smooth Strongly Convex Functions

$$P_i(x) = x - \alpha_i \nabla f_i(x)$$

$$m_i |x - y|^2 \leq (\nabla f_i(x) - \nabla f_i(y))^\top (x - y) \leq L_i |x - y|^2$$

$$0 < \alpha_i < 2m_i / L_i.$$

If the set where all functions achieve the global minimum is non-empty, then the algorithms (1a) and (2a) will compute one of the optimal points.

$$\xi_i(k+1) = \sum\nolimits_j w_{ij} \xi_j(k) - w_{ii} \alpha_i \nabla f_i(\xi_i(k)); \qquad (1a)$$

$$\xi_i(k+1) = v_i(k) - \alpha_i \nabla f_i(v_i(k)), \quad v_i(k) = \sum\nolimits_j w_{ij} \xi_j(k). \qquad (2a)$$

Optimization, but very special: each agent has own cost function, many technical assumptions, no constraints. What about more general cases?

Can we combine the general constrained consensus algorithm with the gradient descent?

Constrained Optimization via Constrained Consensus:

Nedic et al. (IEEE TAC, vol.55, no.4, 2010)

$$\sum_{i=1}^{N} f_i(\xi_i) \to min$$

$$\text{s.t.} \quad \xi_i \in \Xi_i \ \ \forall i = 1, \dots, N.$$

Under certain assumptions, the following algorithm with time-varying step-size parameter finds a global optimum (generally, non-unique)

$$\xi_i(k+1) = P_{\Xi_i}\left[v_i(k) - \alpha_i(k)\nabla f_i(v_i(k))\right], \quad v_i(k) = \sum\nolimits_j w_{ij}\xi_j(k).$$

$$\sum_{k=1}^{\infty} \alpha_i(k) = \infty, \ \sum_{k=1}^{\infty} \alpha_i(k)^2 < \infty$$

Conditions of convergence: double stochasticity of W, bounded (sub)gradients, compact constraint sets (some has been relaxed later). I believe that the method of recurrent inequalities can be extended to cope with such problems (the topic of ongoing research).

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"Altafini's Model": opinion polarization due to Heider's structural balance theory

Averaging-like dynamics over signed graphs

$$x(t+1) = A(t)x(t), \quad t = 0, 1, \ldots$$

$$A(t) = (a_{ij}(t))_{i,j=1}^{n}, \ \sum_{j=1}^{n} |a_{ij}(t)| = 1.$$

- ***n agents*** with ***opinions*** $x_i \in \mathbb{R}$
- ***influence weights*** $a_{ij} \in [-1, 1]$
- Signs of the weights encode ***friendship (+)*** **or** ***enmity (-) and the way of opinion alteration:***

Agent moves her opinion towards her friends' opinions and away from the enemies' opinions

- **Special case:** $a_{ij} \geq 0 \, \forall i, j$ ***DeGroot's model (consensus algorithm)***
- **Generally (under connectivity assumptions):** ***consensus, polarization or asymptotic stability***

Hendrickx 2014; Xia,Cao,Johansson,2016; Liu,Chen,Basar,Belabbas 2017;

- Those behaviors are special cases of the **consensus in modulus** $\lim_{t\to\infty} |x_1(t)| = \ldots = \lim_{t\to\infty} |x_n(t)|$
- **Why does absolute values become synchronous?**

“Altafini’s Model”: a hidden averaging inequality (with a nonnegative solution)

Consensus of absolute values is implied by the inequality, not by Altafini’s dynamics!

$$x(t+1) = A(t)x(t) \implies |x(t+1)| = (|x_i(t+1)|) \leq |A(t)|\,|x(t)|$$

$$|A(t)| = (|a_{ij}(t)|)_{i,j=1}^{n}, \quad \sum_{j=1}^{n} |a_{ij}(t)| = 1.$$

Standard conditions (strict positivity, reciprocity, strong persistent interactions graph) entail

- **the consensus in modulus** $\lim_{k\to\infty} |x_1(k)| = \ldots = \lim_{k\to\infty} |x_N(k)| = m.$
- **vanishing residuals** $|x(k+1)| - |A(k)|\,|x(k)| \xrightarrow[k\to\infty]{} 0$
- **in the non-degenerate case (m>0 for some x(0)): structural balance**

$$a_{ij}(k)\, sgn\, x_i\, sgn\, x_j \geq 0 \quad \forall k \geq k_0$$

- **The agents split into two hostile camps: members of the same camps become friends, members of different camps become enemies in a while**

$$I^+ = \{i : \lim x_i(k) = m > 0\}, I^- = \{i : \lim x_i(k) = -m\}$$

- **The inverse statement is also true: the existence of such camps for large *k* entail polarization of opinions at two value (*+m*) and (*-m*), *m>0* for almost all *x(0)***

Hegselmann-Krause model with "truth-seekers"

Set of agents: $\mathcal{V}$

Opinions: $x_i \in \mathbb{R}^m,\ i \in \mathcal{V}$

$$x_i(t+1) = \frac{1-\lambda_i}{|I_i(x(t))|} \sum_{j \in I_i(x(t))} x_j(t) + \lambda_i \tau, \quad i \in \mathcal{V}.$$

(opinion dynamics)

$$I_i(x) = \{j : \|x_j - x_i\| \le d\} \ni i. \qquad \tau = \text{const} \in \mathbb{R}^m. \qquad \lambda_i \in [0,1].$$

The agents with opinions in the confidence interval of agent *i*

«truth»

The desire to know the truth (attactivity of truth).

- **The weight matrix is state-dependent, and state cannot be found explicitly**
- **Unlike usual consensus algorithms, there is a constant term**
- **The model in fact also converges because it contains an implicit inequality**

$$y_i(t) = \|x_i(t) - \tau\| \ \forall i \implies y(t+1) \le W(x(t))y(t)$$

- **The matrix satisfies the strict positivity and reciprocity condition**

$$w_{ij}(x) = \begin{cases} 1/|I_i(x)|, & \|x_i - x_j\| \le d \\ 0, & \|x_i - x_j\| > d \end{cases} \qquad C^{-1} w_{ji}(x) \le w_{ij}(x) \le C w_{ji}(x)$$

Hegselmann-Krause model with "truth-seekers": convergence

$$x_i(t+1) = \frac{1-\lambda_i}{|I_i(x(t))|} \sum_{j \in I_i(x(t))} x_j(t) + \lambda_i \tau, \quad i \in \mathcal{V}.$$

The convergence theorem

1. **Each solutions converges to a limit** $\exists \bar{x} = \lim_{t\to\infty} x(t).$
2. **Two agents *i* and *j* from the same connectivity component of the persistent graph reach consensus** $\bar{x}_i = \bar{x}_j.$
3. **Agents *i* and *j* from different components of the persistent graphs do not reach consensus, moreover** $\|\bar{x}_i - \bar{x}_j\| \geq d.$
4. **Each truth seeker converges to the truth value** $\lambda_i > 0 \implies \bar{x}_i = \tau.$

The method of recurrent inequalities (P., Calafiore, Cao, 2020) gives the most compact proof. The first proof (Chazelle, 2011) uses non-trivial tools from complex calculus – the method of power series (s-energy).

Some Literature References

Consensus in recurrent and differential inequalities:

- Proskurnikov, A., Cao, M. Differential inequalities in multi-agent coordination and opinion dynamics modeling//Automatica, 2017, 85, 202–210.
- Proskurnikov, A., Cao, M. Modulus consensus in discrete-time signed net- works and properties of special recurrent inequalities//Proc. of IEEE CDC, 2017.
- Proskurnikov, A., Calafiore, G., Cao, M., Recurrent averaging inequalities in multi-agent control and social dynamics modelling//Annual Reviews in Control, 2020, vol. 45
- Proskurnikov, A., Calafiore, G., New Results on Delay Robustness of Consensus Algorithms//Proc. of IEEE CDC 2020

Constrained consensus and distributed algorithms to solve linear equations:

- Nedic, A., Ozdaglar, A., and Parrilo, P. Constrained consensus and optimization in multi-agent networks//IEEE Trans. Autom. Control, 55(4), 2010.
- Mou, S., Liu, J. and Morse, A. A distributed algorithm for solving a linear algebraic equation// IEEE Trans. Autom. Control, vol. 60, no. 11, pp. 2863–2878, 2015
- You, K., Song, S. and Tempo, R. A networked parallel algorithm for solving linear algebraic equations//Proc. of IEEE Conf. Decision and Control, 2016
- Fullmer, D., and Morse, A. A Distributed Algorithm for Computing a Common Fixed Point of a Finite Family of Paracontractions//IEEE Trans. Autom. Control, 2018,vol. 63, no.8

Conclusions and future works

- We have discussed a new tool of MAS theory: recurrent averaging inequalities
- The theory extends (with minor changes) to differential inequalities
- The theory can also be generalized to inequalities with communication delays

- Allow to examine many models and algorithms based on iterative averaging in a uniform and elegant way, shedding light on some assumptions arising in mathematical results (e.g., strong connectivity of communication graphs): fixed point computing, opinion dynamics, containment control (target aggregation) etc.

- Drawback of inequalities: do not admit Lyapunov analysis, hence, convergence rates of the solutions are difficult to find.
- Note however: convergence rate of a standard consensus algorithm is also an open problem, only very conservative estimates are known.

- Topic of ongoing research: marriage of averaging inequalities and distributed optimization
- Another topic: higher-order analogies.


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