Minkowski, Lyapunov, and Bellman:

Inequalities and Equations for Stability and Optimal Control

Saša V. Raković

BIT, Beijing

KTH, Stockholm
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Concepts of Minkowski, Lyapunov, and Bellman in modern control theory.

Hermann Minkowski
Convex Geometry

Aleksandr Lyapunov
Stability Theory

Richard Bellman
Optimal Control

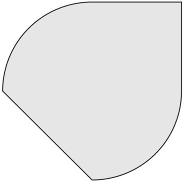
Systematically Fundamental Alternatives to Linear–Quadratic Meme in Controls and Dynamics.

Outline

Part

Minkowski Function

A convex compact subset $\mathcal{S}$ of $\mathfrak{R}^n$ that contains the origin as an interior point is a proper C–set in $\mathfrak{R}^n$.

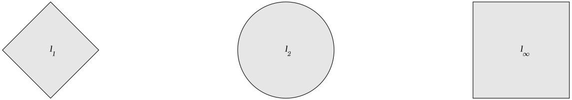


The Minkowski function of a proper C–set $\mathcal{S}$ in $\mathfrak{R}^n$:
$\forall x \in \mathfrak{R}^n,\ g(\mathcal{S}, x) = \min_{\eta}\{\eta\ :\ x \in \eta\mathcal{S},\ \eta \geq 0\}$.

- Continuous and sublinear.
- $g(\mathcal{S}, x) = 0$ for $x = 0$.
- $0 < g(\mathcal{S}, x) < \infty$ for $x \in \mathfrak{R}^n \setminus \{0\}$.

Minkowski Function (Continued)

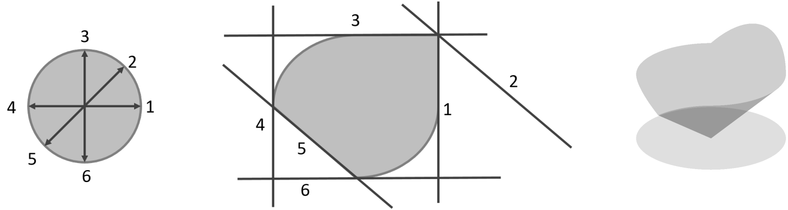
Any vector norm $|\cdot|_{\mathcal{S}}$ on $\mathfrak{R}^n$ is generated by the Minkowski function $g(\mathcal{S},\cdot)$ of a 0–symmetric proper C–set $\mathcal{S}$ in $\mathfrak{R}^n$.
The closed unit norm ball is $\mathcal{S}$.



$\sum_{i=1}^{n} |x_i|$ $\quad$ $\sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ $\quad$ $\max_{i \in \{1,2,\ldots,n\}} |x_i|$

Special cases include $\ell_1,\ \ell_2,\ \ell_\infty, \ldots$ vector norms.

Support Function

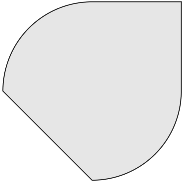


The support function of a nonempty closed convex set $\mathcal{S}$ in $\mathfrak{R}^n$:
$\forall x \in \mathfrak{R}^n,\ h(\mathcal{S}, x) = \sup_s\{x^T s \ :\ s \in \mathcal{S}\}$.

A fundamental **1–to–1** correspondence:

- If $\mathcal{S}$ is a nonempty compact, convex set in $\mathfrak{R}^n$, its support function $h(\mathcal{S}, \cdot)\ :\ \mathfrak{R}^n \to \mathfrak{R}$ is sublinear.
- If $f(\cdot)\ :\ \mathfrak{R}^n \to \mathfrak{R}$ is sublinear function, there is a unique nonempty, compact convex set $\mathcal{S}$ in $\mathfrak{R}^n$ with the support function $f(\cdot)$.

Support Function (Continued)



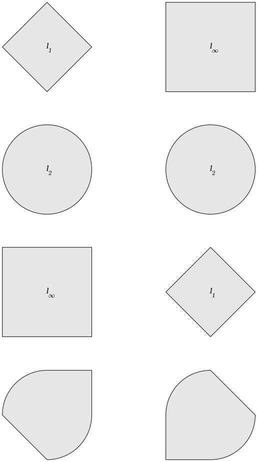
For any nonempty, closed, convex sets $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_2$ in $\mathfrak{R}^n$ and any $M \in \mathfrak{R}^{p\times n}$:

- $\mathcal{S}_1 \subseteq \mathcal{S}_2 \iff \forall x \in \mathfrak{R}^n,\ h(\mathcal{S}_1, x) \leq h(\mathcal{S}_2, x)$.
- $\forall x \in \mathfrak{R}^n,\ h(\mathcal{S}_1 \oplus \mathcal{S}_2, x) = h(\mathcal{S}_1, x) + h(\mathcal{S}_2, x)$.
- $\forall y \in \mathfrak{R}^p,\ h(M\mathcal{S}, y) = h(\mathcal{S}, M^T y)$.

$\mathcal{S}_1 \oplus \mathcal{S}_2 := \{s_1 + s_2 \ : \ s_1 \in \mathcal{S}_1,\ s_2 \in \mathcal{S}_2\}$ is the Minkowski set addition.
$M\mathcal{S} := \{Ms \ : \ s \in \mathcal{S}\}$ is the image of $\mathcal{S}$ under matrix M.

Polarity of Minkowski and Support Functions

The polar set $\mathcal{S}^*$ of a set $\mathcal{S},\ 0 \in \mathcal{S}$ in $\mathfrak{R}^n$: $\mathcal{S}^* := \{x\ :\ \forall y \in \mathcal{S},\ y^T x \leq 1\}$.



Polar set pairs.

Fundamental Polarity Relationships:

If $\mathcal{S}$ is a proper C–set in $\mathfrak{R}^n$, its polar set $\mathcal{S}^*$ is also a proper C–set in $\mathfrak{R}^n$, and its bipolar set $(\mathcal{S}^*)^*$ is $\mathcal{S}$, i.e., $(\mathcal{S}^*)^* = \mathcal{S}$.

The Minkowski function $g(\mathcal{S}, \cdot)$ of a proper C–set $\mathcal{S}$ in $\mathfrak{R}^n$ is equal to the support function $h(\mathcal{S}^*, \cdot)$ of its polar set $\mathcal{S}^*$, i.e., for any proper C–set $\mathcal{S}$ in $\mathfrak{R}^n$,
$\forall x \in \mathfrak{R}^n,\ g(\mathcal{S}, x) = h(\mathcal{S}^*, x)$.

Part

Quadratic Lyapunov Inequality for Linear Dynamics

Linear dynamics: $x^+ = Ax, \quad A \in \mathfrak{R}^{n\times n}$.
Quadratic decrease function: $\ell(x) = x^T Qx, \quad Q \in \mathfrak{R}^{n\times n}$.

Lyapunov inequality: $\forall x \in \mathfrak{R}^n, \quad V(Ax) + \ell(x) \leq V(x)$.
Quadratic Lyapunov function: $V(x) = x^T Px, \quad P \in \mathfrak{R}^{n\times n}$.
Linear matrix inequality: $A^T PA + Q \leq P$.

Fact: Take any $Q \in \mathfrak{R}^{n\times n},\ Q = Q^T > 0$ and any $A \in \mathfrak{R}^{n\times n}$.

There exists a $P \in \mathfrak{R}^{n\times n},\ P = P^T > 0$ such that $A^T PA + Q \leq P$ if and only if A is strictly stable.

Quadratic Lyapunov Equation for Linear Dynamics

Linear dynamics: $x^+ = Ax, \quad A \in \mathfrak{R}^{n\times n}$.
Quadratic decrease function: $\ell(x) = x^T Qx, \quad Q \in \mathfrak{R}^{n\times n}$.

Lyapunov equation: $\forall x \in \mathfrak{R}^n, \quad V(Ax) + \ell(x) = V(x)$.
Quadratic Lyapunov function: $V(x) = x^T Px, \quad P \in \mathfrak{R}^{n\times n}$.
Linear matrix equation: $A^T PA + Q = P$.

Fact: Take any $Q \in \mathfrak{R}^{n\times n}, \; Q = Q^T > 0$ and any $A \in \mathfrak{R}^{n\times n}$.

There exists a $P \in \mathfrak{R}^{n\times n}, \; P = P^T > 0$ such that $A^T PA + Q = P$ if and only if A is strictly stable, in which case $P = \sum_{k=0}^{\infty} (A^k)^T QA^k$ is unique.

Part

Minkowski Algebra of Convex Bodies Essentials

Quadratic Lyapunov Inequality and Equation

Minkowski–Lyapunov Inequality and Equation

Quadratic Bellman Inequality and Equation

Minkowski–Bellman Inequality and Equation

Literature (Published)

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- S. V. Raković. *Minkowski–Lyapunov Functions: Alternative Characterization and Implicit Representation.* Automatica. In Press, Corrected Proof.

Minkowski–Lyapunov Inequality for Linear Dynamics

Linear dynamics:	$x^+ = Ax, \quad A \in \mathfrak{R}^{n\times n}$.
Minkowski decrease function:	$\ell(x) = g(\mathcal{Q}, x), \quad \mathcal{Q} \subset \mathfrak{R}^n$.
Generator set:	$\mathcal{Q}$ is a proper C–set in $\mathfrak{R}^n$.
Lyapunov inequality:	$\forall x \in \mathfrak{R}^n, \quad V(Ax) + \ell(x) \leq V(x)$.
Minkowski–Lyapunov function:	$V(x) = g(\mathcal{P}, x), \quad \mathcal{P} \subset \mathfrak{R}^n$.
Generator set:	$\mathcal{P}$ is a proper C–set in $\mathfrak{R}^n$.

Minkowski–Lyapunov Equation for Linear Dynamics

Linear Dynamics:	$x^{+} = Ax, \quad A \in \mathfrak{R}^{n\times n}.$
Minkowski decrease function:	$\ell(x) = g(\mathcal{Q}, x), \quad \mathcal{Q} \subset \mathfrak{R}^{n}.$
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Minkowski–Lyapunov function:	$V(x) = g(\mathcal{P}, x), \quad \mathcal{P} \subset \mathfrak{R}^{n}.$
Generator set:	$\mathcal{P}$ is a proper C–set in $\mathfrak{R}^{n}$.

Polarity of Stability and Robust Positive Invariance

Minkowski–Lyapunov inequality:
$\forall x \in \mathfrak{R}^n, \quad g(\mathcal{P}, Ax) + g(\mathcal{Q}, x) \leq g(\mathcal{P}, x)$.

Polar form of Minkowski–Lyapunov inequality:
$\forall x \in \mathfrak{R}^n, \quad h(\mathcal{P}^*, Ax) + h(\mathcal{Q}^*, x) \leq h(\mathcal{P}^*, x)$, i.e.,
$\forall x \in \mathfrak{R}^n, \quad h(A^T\mathcal{P}^*, x) + h(\mathcal{Q}^*, x) \leq h(\mathcal{P}^*, x)$, i.e.,
$\forall x \in \mathfrak{R}^n, \quad h(A^T\mathcal{P}^* \oplus \mathcal{Q}^*, x) \leq h(\mathcal{P}^*, x)$.

Equivalent set inclusion:
$A^T\mathcal{P}^* \oplus \mathcal{Q}^* \subseteq \mathcal{P}^*$.

Polar form of equivalent set inclusion:
$\mathcal{P} \subseteq \left(A^T\mathcal{P}^* \oplus \mathcal{Q}^*\right)^*$.

Fundamental Property of Minkowski–Lyapunov Inequality

Minkowski–Lyapunov inequality:
$\forall x \in \mathfrak{R}^n, \quad g(\mathcal{P}, Ax) + g(\mathcal{Q}, x) \leq g(\mathcal{P}, x)$.

Equivalent set inclusion:
$A^T\mathcal{P}^* \oplus \mathcal{Q}^* \subseteq \mathcal{P}^*$.

Polar form of equivalent set inclusion:
$\mathcal{P} \subseteq \left(A^T\mathcal{P}^* \oplus \mathcal{Q}^*\right)^*$.

Theorem III–1. Take any proper C–set $\mathcal{Q}$ in $\mathfrak{R}^n$ and any $A \in \mathfrak{R}^{n\times n}$.

The Minkowski function $x \mapsto g(\mathcal{P}, x)$ of a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ verifies the Minkowski–Lyapunov inequality for $x^+ = Ax$ with the Minkowski decrease function $x \mapsto g(\mathcal{Q}, x)$ if and only if the polar set $\mathcal{P}^*$ of $\mathcal{P}$ verifies set inclusion $A^T\mathcal{P}^* \oplus \mathcal{Q}^* \subseteq \mathcal{P}^*$, or, equivalently, if and only if the set $\mathcal{P}$ verifies set inclusion $\mathcal{P} \subseteq \left(A^T\mathcal{P}^* \oplus \mathcal{Q}^*\right)^*$.

Polar Linear Dynamics and Robust Positive Invariance

Linear Dynamics:	$x^+ = Ax, \quad A \in \mathfrak{R}^{n\times n}$.
Minkowski decrease function:	$\ell(x) = g(\mathcal{Q}, x), \quad \mathcal{Q} \subset \mathfrak{R}^n$.
Generator set:	$\mathcal{Q}$ is a proper C–set in $\mathfrak{R}^n$.
Polar Linear Dynamics:	$z^+ = A^T z + w$.
Polar disturbance:	$w \in \mathcal{W}, \quad \mathcal{W} \subset \mathfrak{R}^n$.
Disturbance set:	$\mathcal{W} := \mathcal{Q}^*$ is a proper C–set in $\mathfrak{R}^n$.

A set $\mathcal{Z}$ is robust positively invariant for $z^+ = A^T z + w$ with $w \in \mathcal{W}$ if and only if $A^T \mathcal{Z} \oplus \mathcal{W} \subseteq \mathcal{Z}$.

A set $\mathcal{Z}$ is the minimal robust positively invariant for $z^+ = A^T z + w$ with $w \in \mathcal{W}$ if and only if $A^T \mathcal{Z} \oplus \mathcal{W} \subseteq \mathcal{Z}$ and $\mathcal{Z}$ is minimal with respect to set inclusion over all (nonempty closed) robust positively invariant sets for $z^+ = A^T z + w$ with $w \in \mathcal{W}$.

Fundamental Property of Minkowski–Lyapunov Inequality

Minkowski–Lyapunov inequality:
$\forall x \in \mathfrak{R}^n, \quad g(\mathcal{P}, Ax) + g(\mathcal{Q}, x) \leq g(\mathcal{P}, x)$.

Equivalent set inclusion:
$A^T \mathcal{P}^* \oplus \mathcal{Q}^* \subseteq \mathcal{P}^*$.

Theorem III–2. Take any proper C–set $\mathcal{Q}$ in $\mathfrak{R}^n$ and any $A \in \mathfrak{R}^{n\times n}$.

The Minkowski function $x \mapsto g(\mathcal{P}, x)$ of a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ verifies the Minkowski–Lyapunov inequality for $x^+ = Ax$ with the Minkowski decrease function $x \mapsto g(\mathcal{Q}, x)$ if and only if the polar set $\mathcal{P}^*$ of $\mathcal{P}$ is a robust positively invariant set for $z^+ = A^T z + w$ with $w \in \mathcal{W}, \ \mathcal{W} := \mathcal{Q}^*$.

Solvability of Minkowski–Lyapunov Inequality

Minkowski–Lyapunov inequality:
$\forall x \in \mathfrak{R}^n, \quad g(\mathcal{P}, Ax) + g(\mathcal{Q}, x) \leq g(\mathcal{P}, x)$.

Equivalent set inclusion:
$A^T\mathcal{P}^* \oplus \mathcal{Q}^* \subseteq \mathcal{P}^*$.

Theorem III–3. Take any proper C–set $\mathcal{Q}$ in $\mathfrak{R}^n$ and any $A \in \mathfrak{R}^{n\times n}$.

There exists a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ whose Minkowski function $g(\mathcal{P}, \cdot)$ verifies the Minkowski–Lyapunov inequality for $x^+ = Ax$ with the Minkowski decrease function $x \mapsto g(\mathcal{Q}, x)$ (equivalently, whose polar set $\mathcal{P}^*$ is a robust positively invariant set for $z^+ = A^T z + w$ with $w \in \mathcal{W},\ \mathcal{W} := \mathcal{Q}^*$) if and only if A is strictly stable.

Minkowski–Lyapunov equation:
$\forall x \in \mathfrak{R}^n, \quad g(\mathcal{P}, Ax) + g(\mathcal{Q}, x) = g(\mathcal{P}, x)$.

Polar form of Minkowski–Lyapunov equation:
$\forall x \in \mathfrak{R}^n, \quad h(\mathcal{P}^*, Ax) + h(\mathcal{Q}^*, x) = h(\mathcal{P}^*, x)$, i.e.,
$\forall x \in \mathfrak{R}^n, \quad h(A^T\mathcal{P}^*, x) + h(\mathcal{Q}^*, x) = h(\mathcal{P}^*, x)$, i.e.,
$\forall x \in \mathfrak{R}^n, \quad h(A^T\mathcal{P}^* \oplus \mathcal{Q}^*, x) = h(\mathcal{P}^*, x)$.

Equivalent set equation:
$A^T\mathcal{P}^* \oplus \mathcal{Q}^* = \mathcal{P}^*$.

Polar form of equivalent set equation:
$\mathcal{P} = \left(A^T\mathcal{P}^* \oplus \mathcal{Q}^*\right)^*$.

Fundamental Property of Minkowski–Lyapunov Equation

Minkowski–Lyapunov equation:
$\forall x \in \mathfrak{R}^n, \quad g(\mathcal{P}, Ax) + g(\mathcal{Q}, x) = g(\mathcal{P}, x)$.

Equivalent set equation:
$A^T \mathcal{P}^* \oplus \mathcal{Q}^* = \mathcal{P}^*$.

Polar form of equivalent set equation:
$\mathcal{P} = \left(A^T \mathcal{P}^* \oplus \mathcal{Q}^*\right)^*$.

Theorem III–4. Take any proper C–set $\mathcal{Q}$ in $\mathfrak{R}^n$ and any $A \in \mathfrak{R}^{n \times n}$.

The Minkowski function $x \mapsto g(\mathcal{P}, x)$ of a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ verifies the Minkowski–Lyapunov equation for $x^+ = Ax$ with the Minkowski decrease function $x \mapsto g(\mathcal{Q}, x)$ if and only if the polar set $\mathcal{P}^*$ of $\mathcal{P}$ solves the fixed point set equation $A^T \mathcal{P}^* \oplus \mathcal{Q}^* = \mathcal{P}^*$, or, equivalently, if and only if the set $\mathcal{P}$ solves the fixed point set equation $\mathcal{P} = \left(A^T \mathcal{P}^* \oplus \mathcal{Q}^*\right)^*$.

Fundamental Property of Minkowski–Lyapunov Equation

Minkowski–Lyapunov equation:
$\forall x \in \mathfrak{R}^n, \quad g(\mathcal{P}, Ax) + g(\mathcal{Q}, x) = g(\mathcal{P}, x)$.

Equivalent set equation:
$A^T \mathcal{P}^* \oplus \mathcal{Q}^* = \mathcal{P}^*$.

Theorem III–5. Take any proper C–set $\mathcal{Q}$ in $\mathfrak{R}^n$ and any $A \in \mathfrak{R}^{n \times n}$.

The Minkowski function $x \mapsto g(\mathcal{P}, x)$ of a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ verifies the Minkowski–Lyapunov equation for $x^+ = Ax$ with the Minkowski decrease function $x \mapsto g(\mathcal{Q}, x)$ if and only if the polar set $\mathcal{P}^*$ of $\mathcal{P}$ is the minimal robust positively invariant set for $z^+ = A^T z + w$ with $w \in \mathcal{W},\ \mathcal{W} := \mathcal{Q}^*$.

Solvability of Minkowski–Lyapunov Equation

Minkowski–Lyapunov equation:
$\forall x \in \mathfrak{R}^n, \quad g(\mathcal{P}, Ax) + g(\mathcal{Q}, x) = g(\mathcal{P}, x)$.

Equivalent set equation:
$A^T \mathcal{P}^* \oplus \mathcal{Q}^* = \mathcal{P}^*$.

Theorem III–6. Take any proper C–set $\mathcal{Q}$ in $\mathfrak{R}^n$ and any $A \in \mathfrak{R}^{n \times n}$.

There exists a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ whose Minkowski function $g(\mathcal{P}, \cdot)$ verifies the Minkowski–Lyapunov equation for $x^+ = Ax$ with the Minkowski decrease function $x \mapsto g(\mathcal{Q}, x)$ (equivalently, whose polar set $\mathcal{P}^*$ is the minimal robust positively invariant set for $z^+ = A^T z + w$ with $w \in \mathcal{W}, \ \mathcal{W} := \mathcal{Q}^*$) if and only if A is strictly stable, in which case $\mathcal{P} = \left(\bigoplus_{k=0}^{\infty} (A^T)^k \mathcal{Q}^*\right)^*$ is unique.

Theoretical Ramification.
The stability analysis via the Minkowski–Lyapunov functions for $x^+ = Ax$ with the Minkowski decrease function $x \mapsto g(\mathcal{Q}, x)$ is conceptually identical to the robust positive invariance analysis, over the space of proper C–sets, for $z^+ = A^T z + w$ with $w \in \mathcal{W},\ \mathcal{W} := \mathcal{Q}^*$.

Computational Ramification.
The computation of the Minkowski–Lyapunov functions for $x^+ = Ax$ with the Minkowski decrease function $x \mapsto g(\mathcal{Q}, x)$ is in an essential one–to–one correspondence with the computation of proper C robust positively invariant sets for $z^+ = A^T z + w$ with $w \in \mathcal{W},\ \mathcal{W} := \mathcal{Q}^*$.

Part

Quadratic Bellman Inequality for Linear Systems

Linear system:	$x^+ = Ax + Bu,$ $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}.$
Quadratic stage cost function:	$\ell(x, u) = x^T Qx + u^T Ru,$ $(Q, R) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{m\times m}.$
Quadratic Bellman function: (a.k.a. cost–to–go function)	$V(x) = x^T Px,$ $P \in \mathfrak{R}^{n\times n}.$
Bellman functional dynamics: (Value function) (Optimizer function)	$\forall x \in \mathfrak{R}^n,$ $V^+(x) := \min_u V(Ax + Bu) + \ell(x, u).$ $u^+(x) := \arg\min_u V(Ax + Bu) + \ell(x, u).$
Quadratic Bellman inequality:	$\forall x \in \mathfrak{R}^n, \quad V^+(x) \leq V(x).$

Quadratic Bellman Equation for Linear Systems

Linear system:	$x^+ = Ax + Bu$, $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.
Quadratic stage cost function:	$\ell(x, u) = x^T Qx + u^T Ru$, $(Q, R) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{m\times m}$.
Quadratic Bellman function: (a.k.a. cost–to–go function)	$V(x) = x^T Px$, $P \in \mathfrak{R}^{n\times n}$.
Bellman functional dynamics: (Value function) (Optimizer function)	$\forall x \in \mathfrak{R}^n$, $V^+(x) := \min_u V(Ax + Bu) + \ell(x, u)$. $u^+(x) := \arg\min_u V(Ax + Bu) + \ell(x, u)$.
Quadratic Bellman equation:	$\forall x \in \mathfrak{R}^n, \quad V^+(x) = V(x)$.

Essential Equivalence of Bellman and Riccati Dynamics

Fact. Take any matrices $Q \in \mathfrak{R}^{n\times n},\ Q = Q^T > 0$, $R \in \mathfrak{R}^{m\times m},\ R = R^T > 0$ and $P \in \mathfrak{R}^{n\times n},\ P = P^T > 0$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

The Bellman functional dynamics

$$\begin{aligned} \forall x \in \mathfrak{R}^n, \quad V^+(x) &= \min_u V(Ax + Bu) + \ell(x, u) \\ &= x^T P^+ x, \text{ and} \\ u^+(x) &= \arg\min_u V(Ax + Bu) + \ell(x, u) \\ &= K^+ x \end{aligned}$$

is essentially equivalent to the Riccati matrix dynamics

$$\begin{aligned} &P^+ = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA, \text{ and} \\ &K^+ = -(R + B^T PB)^{-1} B^T PA. \end{aligned}$$

Finite Horizon Linear Quadratic Regulator

Fact. Take any matrices $Q \in \mathfrak{R}^{n \times n},\ Q = Q^T > 0$, $R \in \mathfrak{R}^{m \times m},\ R = R^T > 0$ and $P \in \mathfrak{R}^{n \times n},\ P = P^T > 0$ and any matrix pair $(A, B) \in \mathfrak{R}^{n \times n} \times \mathfrak{R}^{n \times m}$.

Iteration of the Bellman functional dynamics, or, equivalently, the Riccati matrix dynamics

$$
\begin{aligned}
\forall x \in \mathfrak{R}^n, \quad & V^+(x) = x^T P^+ x, \text{ and} \\
& u^+(x) = K^+ x, \text{ with} \\
& P^+ = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA, \text{ and} \\
& K^+ = -(R + B^T PB)^{-1} B^T PA
\end{aligned}
$$

yields **the finite horizon linear quadratic regulator**.

Fundamental Property of Quadratic Bellman Inequality

Fact. Take any matrices $Q \in \mathfrak{R}^{n\times n},\ Q = Q^T > 0$, $R \in \mathfrak{R}^{m\times m},\ R = R^T > 0$ and $P \in \mathfrak{R}^{n\times n},\ P = P^T > 0$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

The quadratic Bellman inequality

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) \leq V(x),$$

is essentially equivalent to the Riccati matrix inequality

$$P^+ \leq P \text{ i.e., } Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA \leq P.$$

Solvability of Quadratic Bellman Inequality

Fact. Take any matrices $Q \in \mathfrak{R}^{n\times n},\ Q = Q^T > 0$, $R \in \mathfrak{R}^{m\times m},\ R = R^T > 0$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

There exists a $P \in \mathfrak{R}^{n\times n},\ P = P^T > 0$ verifying the quadratic Bellman inequality, or, equivalently, the Riccati matrix inequality

$$Q + A^T PA - A^T PB(R + B^T PB)^{-1}B^T PA \leq P$$

if and only if (A, B) is strictly stabilizable.

Analogous conclusions to this and previous facts hold when $\ell(x, u) = (x, u)^T C(x, u)$ for any $C \in \mathfrak{R}^{(n+m)\times(n+m)},\ C = C^T > 0$.

Fundamental Property of Quadratic Bellman Equation

Fact. Take any matrices $Q \in \mathfrak{R}^{n\times n},\ Q = Q^T > 0$, $R \in \mathfrak{R}^{m\times m},\ R = R^T > 0$ and $P \in \mathfrak{R}^{n\times n},\ P = P^T > 0$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

The quadratic Bellman equation, i.e., the fixed point of Bellman functional dynamics

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) = V(x)$$

is essentially equivalent to the Riccati matrix equation, i.e., the fixed point of the Riccati matrix dynamics

$$P^+ = P \text{ i.e., } Q + A^T PA - A^T PB(R + B^T PB)^{-1}B^T PA = P.$$

Infinite Horizon Linear Quadratic Regulator

Fact. Take any matrices $Q \in \mathfrak{R}^{n\times n},\ Q = Q^T > 0$, $R \in \mathfrak{R}^{m\times m},\ R = R^T > 0$ and $P \in \mathfrak{R}^{n\times n},\ P = P^T > 0$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

The quadratic Bellman equation, or, equivalently, the Riccati matrix equation

$$
\begin{aligned}
\forall x \in \mathfrak{R}^n, \quad & V^+(x) = V(x) = x^T P x, \text{ and} \\
& u^+(x) = u(x) = Kx, \text{ with} \\
& P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA, \text{ and} \\
& K = -(R + B^T PB)^{-1} B^T PA
\end{aligned}
$$

yields **the infinite horizon linear quadratic regulator**.

Solvability of Quadratic Bellman Equation

Fact. Take any matrices $Q \in \mathfrak{R}^{n\times n},\ Q = Q^T > 0$,
$R \in \mathfrak{R}^{m\times m},\ R = R^T > 0$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

There exists a $P \in \mathfrak{R}^{n\times n},\ P = P^T > 0$ verifying the quadratic Bellman equation, or equivalently, the Riccati matrix equation

$$Q + A^T PA - A^T PB(R + B^T PB)^{-1}B^T PA = P$$

if and only if (A, B) is strictly stabilizable, in which case P is unique.

Analogous conclusions to this and previous facts hold when
$\ell(x, u) = (x, u)^T C(x, u)$ for any $C \in \mathfrak{R}^{(n+m)\times(n+m)},\ C = C^T > 0$.

Part

Literature (Published)

- S. V. Raković. *The Minkowski–Bellman Equation.* Technical report, Beijing Institute of Technology, Beijing, China, 2019. Publicly available at ResearchGate and ArXiv.
- S. V. Raković. *Minkowski–Bellman Inequality and Equation.* Automatica. 125: 109435, 2021.
- S. V. Raković. *Control Minkowski–Lyapunov Functions.* Automatica. 128: 109598, 2021.

Minkowski–Bellman Inequality for Linear Systems

Linear system:	$x^+ = Ax + Bu$, $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.
Minkowski stage cost function:	$\ell(x, u) = g(\mathcal{C}, (x, u))$.
Generator set	$\mathcal{C}$ is a proper C–set in $\mathfrak{R}^{n+m}$.
Minkowski–Bellman function:	$V(x) = g(\mathcal{P}, x)$.
Generator set:	$\mathcal{P}$ is a proper C–set in $\mathfrak{R}^n$.
Bellman functional dynamics:	$\forall x \in \mathfrak{R}^n$,
(Value function)	$V^+(x) := \min_u V(Ax + Bu) + \ell(x, u)$.
(Optimizer map)	$u^+(x) := \arg\min_u V(Ax + Bu) + \ell(x, u)$.
Minkowski–Bellman inequality:	$\forall x \in \mathfrak{R}^n, \quad V^+(x) \le V(x)$.

Minkowski–Bellman Equation for Linear Systems

Linear system:	$x^+ = Ax + Bu$,
	$(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.
Minkowski stage cost function:	$\ell(x, u) = g(\mathcal{C}, (x, u))$.
Generator set	$\mathcal{C}$ is a proper C–set in $\mathfrak{R}^{n+m}$.
Minkowski–Bellman function:	$V(x) = g(\mathcal{P}, x)$.
Generator set:	$\mathcal{P}$ is a proper C–set in $\mathfrak{R}^n$.
Bellman functional dynamics:	$\forall x \in \mathfrak{R}^n$,
(Value function)	$V^+(x) := \min_u V(Ax + Bu) + \ell(x, u)$.
(Optimizer map)	$u^+(x) := \arg\min_u V(Ax + Bu) + \ell(x, u)$.
Minkowski–Bellman equation:	$\forall x \in \mathfrak{R}^n, \quad V^+(x) = V(x)$.

What is $J(x,u) = V(Ax+Bu) + \ell(x,u)$?

$$
\begin{aligned}
\forall (x,u) \in \mathfrak{R}^n \times \mathfrak{R}^m, \quad J(x,u) &= V(Ax+Bu) + \ell(x,u) \\
&= g(\mathcal{P}, Ax+Bu) + g(\mathcal{C}, (x,u)) \\
&= h(\mathcal{P}^*, Ax+Bu) + h(\mathcal{C}^*, (x,u)) \\
&= h((A\ B)^T \mathcal{P}^*, (x,u)) + h(\mathcal{C}^*, (x,u)) \\
&= h((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*, (x,u)) \\
&= g(((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*, (x,u)) \\
&= g(\mathcal{T}^+, (x,u)).
\end{aligned}
$$

Characterization of $J(x, u) = V(Ax + Bu) + \ell(x, u)$

$$\begin{aligned}\forall (x,u) \in \mathfrak{R}^n \times \mathfrak{R}^m, \quad J(x,u) &= V(Ax+Bu) + \ell(x,u) \\ &= g(\mathcal{P}, Ax+Bu) + g(\mathcal{C}, (x,u)) \\ &= g(\mathcal{T}^+, (x,u)).\end{aligned}$$

Theorem V–1. Take any proper C–sets $\mathcal{P}$ and $\mathcal{C}$ in $\mathfrak{R}^n$ and $\mathfrak{R}^{n+m}$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

$$\forall (x,u) \in \mathfrak{R}^n \times \mathfrak{R}^m, \quad J(x,u) = V(Ax+Bu) + \ell(x,u) = g(\mathcal{T}^+, (x,u)),$$

for a proper C–set $\mathcal{T}^+ := ((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*$ in $\mathfrak{R}^{n+m}$.

What is $V^+(x) = \min_u g(\mathcal{T}^+, (x, u))$?

$$\forall (x, u) \in \mathfrak{R}^n \times \mathfrak{R}^m, \quad J(x, u) = g(\mathcal{T}^+, (x, u)).$$

Let $\mathcal{P}^+$ be the projection of $\mathcal{T}^+$ onto $\mathfrak{R}^n$, i.e., $\mathcal{P}^+ := (I \;\; O)\mathcal{T}^+$.
First,

$$\forall x \in \mathfrak{R}^n, \quad x \in g(\mathcal{P}^+, x)\mathcal{P}^+ \text{ and } \exists v \in \mathfrak{R}^m : \; (x, v) \in g(\mathcal{P}^+, x)\mathcal{T}^+, \text{ i.e.,}$$
$$\min_u g(\mathcal{T}^+, (x, u)) \leq g(\mathcal{P}^+, x).$$

Second,

$$\forall x \in \mathfrak{R}^n, \quad \exists v \in \mathfrak{R}^m : \; (x, v) \in \left(\min_u g(\mathcal{T}^+, (x, u))\right) \mathcal{T}^+, \text{ i.e.,}$$
$$x \in \left(\min_u g(\mathcal{T}^+, (x, u))\right) \mathcal{P}^+, \text{ i.e.,}$$
$$g(\mathcal{P}^+, x) \leq \min_u g(\mathcal{T}^+, (x, u)).$$

Hence,

$$\forall x \in \mathfrak{R}^n, \quad \min_u g(\mathcal{T}^+, (x, u)) = g(\mathcal{P}^+, x).$$

What is $u^+(x) = \arg\min_u g(\mathcal{T}^+, (x, u))$?

$$\forall (x, u) \in \mathfrak{R}^n \times \mathfrak{R}^m, \quad J(x, u) = g(\mathcal{T}^+, (x, u)).$$

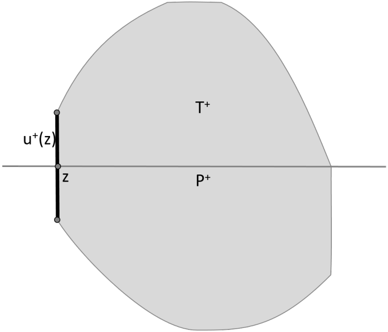
With $\mathcal{P}^+$ equal to the projection of $\mathcal{T}^+$ onto $\mathfrak{R}^n$, i.e., $\mathcal{P}^+ := (I \;\; O)\mathcal{T}^+$,

$$\forall x \in \mathfrak{R}^n, \quad \min_u g(\mathcal{T}^+, (x, u)) = g(\mathcal{P}^+, x).$$

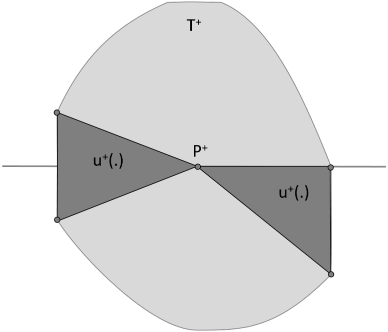
Hence,

$$\begin{aligned} \forall x \in \mathfrak{R}^n, \quad u^+(x) &= \arg\min_u g(\mathcal{T}^+, (x, u)) \\ &= \{u \in \mathfrak{R}^m \ : \ g(\mathcal{T}^+, (x, u)) \leq g(\mathcal{P}^+, x)\} \\ &= \{u \in \mathfrak{R}^m \ : \ (x, u) \in g(\mathcal{P}^+, x)\mathcal{T}^+\}. \end{aligned}$$

What are $V^+(x) = \min_u J(x,u)$ and $u^+(x) = \arg V^+(x)$?



Pointwise Optimization



Parameterwise Optimization

$$\forall (x,u) \in \mathfrak{R}^n \times \mathfrak{R}^m, \quad J(x,u) = g(\mathcal{T}^+, (x,u)).$$

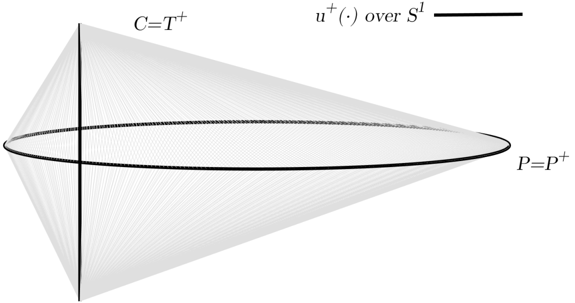
Theorem V–2. Take any proper C–sets $\mathcal{P}$ and $\mathcal{C}$ in $\mathfrak{R}^n$ and $\mathfrak{R}^{n+m}$ and any matrix pair $(A,B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

$$\begin{aligned} \forall x \in \mathfrak{R}^n, \quad V^+(x) &= \min_u g(\mathcal{T}^+, (x,u)) \\ &= g(\mathcal{P}^+, x), \text{ and} \\ u^+(x) &= \arg\min_u g(\mathcal{T}^+, (x,u)) \\ &= \{u \in \mathfrak{R}^m \ : \ (x,u) \in g(\mathcal{P}^+, x)\mathcal{T}^+\}, \end{aligned}$$

for a proper C–set $\mathcal{P}^+ := (I \ O)\mathcal{T}^+$ in $\mathfrak{R}^n$, with $\mathcal{T}^+ := ((A \ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*$.

Geometry of Minkowski–Bellman Dynamics

$A = O, \quad B = O$ with $\mathcal{P}^+ = \mathcal{P} = \mathcal{B}_2^2$ and $\mathcal{T}^+ = \mathcal{C}$.



An example from a private communication with Zvi Artstein.

What are properties of $u^+(x) = \arg\min_u g(\mathcal{T}^+, (x,u))$?

$\mathcal{P}^+ := (I\ O)\mathcal{T}^+$ with $\mathcal{T}^+ := ((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*$ and

$$\forall x \in \mathfrak{R}^n, \quad u^+(x) = \{u \in \mathfrak{R}^m \ :\ (x,u) \in g(\mathcal{P}^+, x)\mathcal{T}^+\}.$$

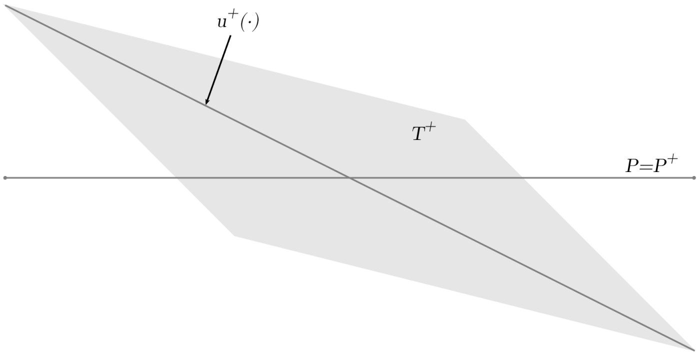
Theorem V–3. Take any proper C–sets $\mathcal{P}$ and $\mathcal{C}$ in $\mathfrak{R}^n$ and $\mathfrak{R}^{n+m}$ and any matrix pair $(A,B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.
The optimizer map $u^+(\cdot)\ :\ \mathfrak{R}^n \rightrightarrows \mathfrak{R}^m$ is

- Positively homogeneous of the first degree.
  (i.e., $\forall x \in \mathfrak{R}^n, \quad \forall \eta \in \mathfrak{R}_{\geq 0}, \quad u^+(\eta x) = \eta u^+(x)$.)
- Compact– and convex–valued.
- Locally bounded.
- Outer semicontinuous.

$u^+(\cdot)$ is a positively homogeneous of the first degree and continuous function when it is single valued (e.g., when $\mathcal{T}^+$ is strictly convex).

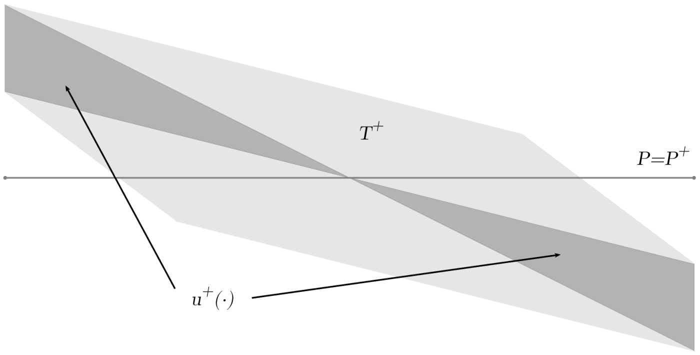
Academic Example with Single Valued Optimizer

$A = 1, \quad B = 1$ with
$\mathcal{P}^+ = \mathcal{P} = [-1, 1]$ and $\mathcal{C} = [-1, 1] \times [-1, 1]$ and
$\mathcal{T}^+ = \text{convh}(\{(1, -1), (\frac{1}{3}, \frac{1}{3}), (-1, 1), (-\frac{1}{3}, -\frac{1}{3})\})$.



Academic Example with Set Valued Optimizer

$A = 1, \quad B = 1$ with
$\mathcal{P}^+ = \mathcal{P} = [-1, 1]$ and $\mathcal{C} = [-2, 2] \times [-1, 1]$ and
$\mathcal{T}^+ = \mathrm{convh}(\{(-1, 1), (\frac{1}{2}, \frac{1}{4}), (1, -\frac{1}{2}), (1, -1), (-\frac{1}{2}, -\frac{1}{4}), (-1, \frac{1}{2})\})$.



Essential Equivalence of Bellman and Minkowski Dynamics

Theorem V–4. Take any proper C–sets $\mathcal{P}$ and $\mathcal{C}$ in $\mathfrak{R}^n$ and $\mathfrak{R}^{n+m}$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

The Bellman functional dynamics

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) = \min_u g(\mathcal{P}, Ax + Bu) + g(\mathcal{C}, (x, u)), \text{ and}$$
$$u^+(x) = \arg\min_u g(\mathcal{P}, Ax + Bu) + g(\mathcal{C}, (x, u))$$

is essentially equivalent to **the Minkowski set dynamics**

$$\mathcal{T}^+ = ((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*, \text{ and}$$
$$\mathcal{P}^+ = (I\ O)\mathcal{T}^+.$$

In particular,

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) = g(\mathcal{P}^+, x), \text{ and}$$
$$u^+(x) = \{u \in \mathfrak{R}^m \ :\ (x, u) \in g(\mathcal{P}^+, x)\mathcal{T}^+\}.$$

Finite Horizon Linear Minkowski Regulator

Theorem V–5. Take any proper C–sets $\mathcal{P}$ and $\mathcal{C}$ in $\mathfrak{R}^n$ and $\mathfrak{R}^{n+m}$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

Iteration of the Bellman functional dynamics, or, equivalently, the Minkowski set dynamics

$$\begin{aligned}\forall x \in \mathfrak{R}^n, \quad & V^+(x) = g(\mathcal{P}^+, x), \text{ and}\\ & u^+(x) = \{u \in \mathfrak{R}^m \ : \ (x, u) \in g(\mathcal{P}^+, x)\mathcal{T}^+\}, \text{ with}\\ & \mathcal{T}^+ = ((A\ B)^T\mathcal{P}^* \oplus \mathcal{C}^*)^*, \text{ and}\\ & \mathcal{P}^+ = (I\ \ O)\mathcal{T}^+\end{aligned}$$

yields **the finite horizon linear Minkowski regulator**.

Fundamental Property of Minkowski–Bellman Inequality

Minkowski–Bellman inequality

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) \leq V(x) \quad \text{i.e.,} \quad g(\mathcal{P}^+, x) \leq g(\mathcal{P}, x).$$

Equivalent set inclusion of related generator sets

$$\mathcal{P} \subseteq \mathcal{P}^+ \quad \text{i.e.,} \quad \mathcal{P} \subseteq (I\ O)((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*.$$

Theorem V–6. Take any proper C–set $\mathcal{C}$ in $\mathfrak{R}^{n+m}$ and any matrix pair $(A, B) \in \mathfrak{R}^{n \times n} \times \mathfrak{R}^{n \times m}$.

The Minkowski function $x \mapsto g(\mathcal{P}, x)$ of a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ verifies the Minkowski–Bellman inequality if and only if its generator set $\mathcal{P}$ verifies set inclusion $\mathcal{P} \subseteq (I\ O)((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*$.

Minkowski–Bellman inequality

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) \leq V(x) \quad \text{i.e.,} \quad g(\mathcal{P}^+, x) \leq g(\mathcal{P}, x).$$

Equivalent set inclusion of related generator sets

$$\mathcal{P} \subseteq \mathcal{P}^+ \quad \text{i.e.,} \quad \mathcal{P} \subseteq (I \ O)((A \ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*.$$

Theorem V–7. Take any proper C–set $\mathcal{C}$ in $\mathfrak{R}^{n+m}$ and any matrix pair $(A, B) \in \mathfrak{R}^{n \times n} \times \mathfrak{R}^{n \times m}$.

There exists a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ whose Minkowski function $x \mapsto g(\mathcal{P}, x)$ verifies the Minkowski–Bellman inequality if and only if the matrix pair (A, B) is strictly stabilizable.

Fundamental Property of Minkowski–Bellman Equation

Theorem V–8. Take any proper C–sets $\mathcal{P}$ and $\mathcal{C}$ in $\mathfrak{R}^n$ and $\mathfrak{R}^{n+m}$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

The Minkowski–Bellman equation, i.e., the fixed point of Bellman functional dynamics

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) = V(x) \quad \text{i.e.,} \quad g(\mathcal{P}^+, x) = g(\mathcal{P}, x)$$

is essentially equivalent to the fixed point of the Minkowski set dynamics

$$\mathcal{P} = \mathcal{P}^+ \quad \text{i.e.,} \quad \mathcal{P} = (I\ O)((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*.$$

In particular, the Minkowski function $x \mapsto g(\mathcal{P}, x)$ of a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ verifies the Minkowski–Bellman equation if and only if its generator set $\mathcal{P}$ verifies the set equation $\mathcal{P} = (I\ O)((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*$.

Theorem V–9. Take any proper C–sets $\mathcal{P}$ and $\mathcal{C}$ in $\mathfrak{R}^n$ and $\mathfrak{R}^{n+m}$ and any matrix pair $(A, B) \in \mathfrak{R}^{n\times n} \times \mathfrak{R}^{n\times m}$.

The Minkowski–Bellman equation

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) = V(x) \quad \text{i.e.,} \quad g(\mathcal{P}^+, x) = g(\mathcal{P}, x)$$

or, equivalently, the fixed point of the Minkowski set dynamics

$$\mathcal{P} = \mathcal{P}^+ \quad \text{i.e.,} \quad \mathcal{P} = (I\ O)((A\ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*$$

yields **the infinite horizon linear Minkowski regulator**.

Minkowski–Bellman equation

$$\forall x \in \mathfrak{R}^n, \quad V^+(x) = V(x) \quad \text{i.e.,} \quad g(\mathcal{P}^+, x) = g(\mathcal{P}, x).$$

Equivalent fixed point set equation

$$\mathcal{P} = \mathcal{P}^+ \quad \text{i.e.,} \quad \mathcal{P} = (I \ O)((A \ B)^T \mathcal{P}^* \oplus \mathcal{C}^*)^*.$$

Theorem V–10. Take any proper C–set $\mathcal{C}$ in $\mathfrak{R}^{n+m}$ and any matrix pair $(A, B) \in \mathfrak{R}^{n \times n} \times \mathfrak{R}^{n \times m}$.

There exists a proper C–set $\mathcal{P}$ in $\mathfrak{R}^n$ whose Minkowski function $x \mapsto g(\mathcal{P}, x)$ verifies the Minkowski–Bellman equation if and only if the matrix pair (A, B) is strictly stabilizable, in which case $\mathcal{P}$ is unique.

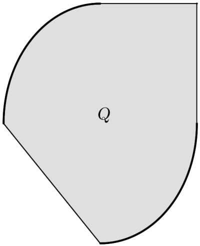
(Existence through Blaschke Selection Theorem.)
(Uniqueness through Minkowski set dynamics asymptotic stability.)

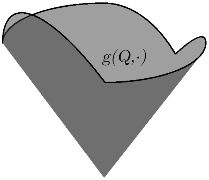
Final Academic Example: Setting

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

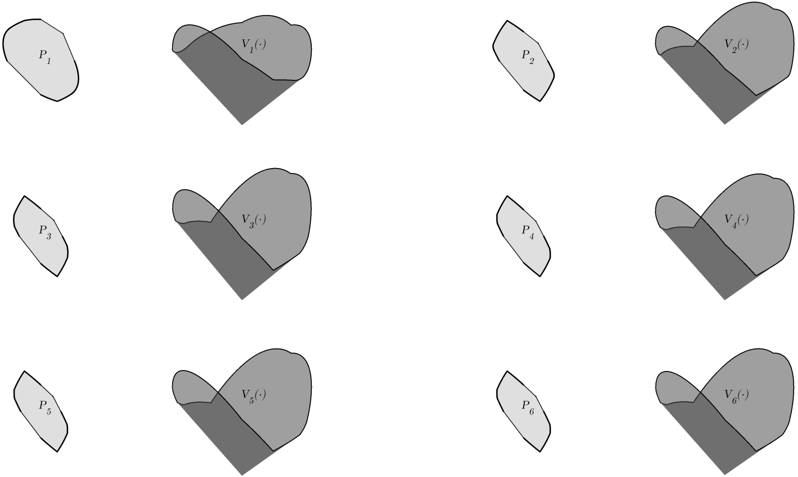
$$\mathcal{C} = ((I\ O)^T \mathcal{Q}^* \oplus (0\ I)^T \mathcal{R}^*)^* \quad \text{with} \quad \mathcal{R} = \mathcal{R}^* = [-1, 1] \quad \text{and}$$

$$g(\mathcal{Q}, (\xi_1, \xi_2)) = \begin{cases} \ell_2((\xi_1, \xi_2)) & \text{when} \quad \xi_1 \xi_2 \leq 0 \\ \ell_\infty((\xi_1, \xi_2)) & \text{when} \quad \xi_1 \geq 0 \quad \text{and} \quad \xi_2 \geq 0 \\ \ell_1((\xi_1, \xi_2)) & \text{when} \quad \xi_1 \leq 0 \quad \text{and} \quad \xi_2 \leq 0 \end{cases}$$



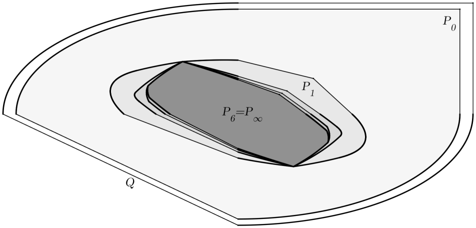


Final Academic Example: Iterates

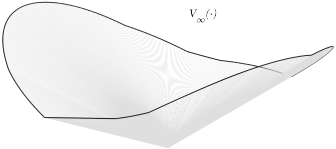


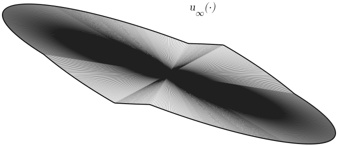
Iterates of Minkowski set dynamics and Bellman functional dynamics

Final Academic Example: Fixed Point



Iterates of Minkowski set dynamics





Fixed Point Value Function and Optimizer Map

Minkowski Algebra for Proper C–Polytopes

For a proper C–polytope $\mathcal{P} = \{x \ : \ \forall i \in \mathcal{I}_\mathcal{P}, \ \phi_i^T x \leq 1\}$:
$\mathcal{P}^* = \text{convh}(\{\phi_i \ : \ i \in \mathcal{I}_\mathcal{P}\})$ and
$(A \ B)^T \mathcal{P}^* = \text{convh}(\{(A^T \phi_i, B^T \phi_i) \ : \ i \in \mathcal{I}_\mathcal{P}\})$.

For a proper C–polytope $\mathcal{C} = \{(x, u) \ : \ \forall i \in \mathcal{I}_\mathcal{C}, \ \theta_i^T x + \vartheta_i^T u \leq 1\}$:
$\mathcal{C}^* = \text{convh}(\{(\theta_i, \vartheta_i) \ : \ i \in \mathcal{I}_\mathcal{C}\})$.

Hence,
$(A \ B)^T \mathcal{P}^* \oplus \mathcal{C}^* = \text{convh}(\{(A^T \phi_i + \theta_j, B^T \phi_i + \vartheta_j \ : \ (i, j) \in \mathcal{I}_\mathcal{P} \times \mathcal{I}_\mathcal{C}\})$,
i.e., (possibly redundant) representation of $\mathcal{T}^+$ is
$\mathcal{T}^+ = \{(x, u) \ : \ \forall (i, j) \in \mathcal{I}_\mathcal{P} \times \mathcal{I}_\mathcal{C}, \ (A^T \phi_i + \theta_j)^T x + (B^T \phi_i + \vartheta_j)^T u \leq 1\}$.

Explicit Characterization for Proper C–Polytopes

$\mathcal{T}^+ = \{(x, u) \ : \ \forall i \in \mathcal{I}_{\mathcal{T}^+}, \ \beta_i^T x + \gamma_i^T u \leq 1\}$, and
$\mathcal{P}^+ = (I \ O)\mathcal{T}^+$ so $\mathcal{P}^+ = \{x \ : \ \forall i \in \mathcal{I}_{\mathcal{P}^+}, \ \alpha_i^T x \leq 1\}$.

Value function $V^+(x) = g(\mathcal{P}^+, x)$ is given, for all $x \in \mathfrak{R}^n$, by

$g(\mathcal{P}^+, x) = \alpha_i^T x$ when $x \in \mathbb{P}_i^+$ with
$\forall i \in \mathcal{I}_{\mathcal{P}^+}, \quad \mathbb{P}_i^+ = \{x \ : \ \forall j \in \mathcal{I}_{\mathcal{P}^+} \setminus \{i\}, \quad (\alpha_j - \alpha_i)^T x \leq 0\}$.

Optimizer Map $u^+(x) = \{u \ : \ (x, u) \in g(\mathcal{P}^+, x)\mathcal{T}^+\}$ is given, for all $x \in \mathfrak{R}^n$, by

$u^+(x) = \{u \ : \ \forall j \in \mathcal{I}_{\mathcal{T}^+}, \quad \gamma_j^T u \leq (\alpha_i - \beta_j)^T x\}$ when $x \in \mathbb{P}_i^+$.

Summary for Proper C–Polytopes

Additional and refined properties (as applicable):

$V^+(\cdot)$ is, finitely many pieces, piecewise linear.

$u^+(\cdot)$ is polytopic–valued, its graph is a finite union of polyhedral cones, and it is Lipschitz continuous with respect to the Hausdorff distance.

Words of Caution:

Fixed point value function **CAN NOT** be a priori guaranteed to be a Minkowski function of a proper C–polytope.

Additional and refined properties **CAN NOT** be a priori guaranteed to hold for fixed point value function and its optimizer.

Theoretical and Computational Ramifications

Theoretical Ramification.
Polarity related theoretical ramifications are somewhat more involved.

Computational Ramification.
Computations via Bellman functional dynamics or Minkowski set dynamics are essentially identical/equivalent in a well defined sense.

Utility.
"There is nothing so practical as a good theory." (Kurt Lewin.)

Generalizations with Literature (Published and in Progress)

✓ **Robust Minkowski–Lyapunov Inequality and Equation**

- S. V. Raković. *Robust Minkowski–Lyapunov Functions.* Automatica. 120: 109168, 2021.
- S. V. Raković. *The Robust Minkowski–Lyapunov Equation.* IEEE–TAC. In Press, Corrected Proof.

✓ **Control and Robust Control Minkowski–Lyapunov Inequalities**

- S. V. Raković. *Control Minkowski–Lyapunov Functions.* Automatica. 128: 109598, 2021.
- S. V. Raković. *Robust Control Minkowski–Lyapunov Functions.* Automatica. 125: 109437, 2021.

✓ **Robust Minkowski–Bellman Inequality and Equation**

- S. V. Raković and M. Jaćimović. *Robust Linear Minkowski Regulator.* In Revision.

Closing Message

Rudolf Kalman

Message from Rudolf Kalman's Plenary Talk (at the 17^{th} IFAC World Congress 2008)
"... further significant advances when $a \in \mathcal{A}$ becomes standard in control .... "

- Set–valued analysis and calculus for control available since 1960
- Increasing popularity in contemporary control theory and applications
- Conceptually powerful and topologically flexible
- Nontrivial computational issues and challenges

Rather peculiarly, developed results I presented to you are novel!

Credits

Zvi Artstein

for constructive feedback on numerous occasions.

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for helpful initial discussion.

Thank you! Any questions?